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Singular equivalences and Auslander-Reiten conjecture

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ABSTRACT

Auslander-Reiten conjecture, which says that an Artin algebra does not have any non-projective generator with vanishing self-extensions in all positive degrees, is shown to be invariant under certain singular equivalences induced by adjoint pairs, which occur often in matrix algebras, recollements and change of rings. Accordingly, several reduction methods are established to study this conjecture.

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1. Introduction

In the representation theory of algebras, the celebrated Nakayama conjecture (NC for short) states that an Artin algebra A is self-injective provided that all terms in a minimal injective resolution of A are projective. In [24], Müller restated the Nakayama conjecture and proved that the Nakayama conjecture holds for all algebras if and only if for each algebra A, a generator-cogenerator M of A with $\mathrm{Ext}^i_A(M,M)$ vanishing for all $i>0$ is necessarily projective. In [3], Auslander and Reiten proposed a conjecture which is an analogue of Müller's theorem.

Auslander-Reiten conjecture (ARC): *A generator (may not be a cogenerator) M of A with $\mathrm{Ext}^i_A(M,M)=0$ for all $i>0$ must be projective, or equivalently, a finitely generated A-module X satisfying $\mathrm{Ext}^i_A(X, X\oplus A)=0$ for all $i>0$ is necessarily projective.*

ARC is closely connected with NC and several famous conjectures in the representation theory of Artin algebras. For example, let A be an Artin algebra.

Finitistic dimension conjecture (FDC): *The finitistic dimension of A (the supremum of the projective dimensions of finitely generated A-modules with finite projective dimension) is finite.*

Strong Nakayama conjecture (SNC): *For each nonzero finitely generated A-module M, there is an integer $n\geqslant 0$ such that $\mathrm{Ext}^n_A(M,A)\neq 0$* [13].

Generalized Nakayama conjecture (GNC): *Every indecomposable injective A-module occurs as a direct summand of a term in a minimal injective resolution of ${}_AA$* [3].

All these conjectures are widely open. The relationship among them is stated as follows. For more details, we refer the readers to [3], [40, Theorem 3.4.3].

- For an individual algebra, FDC $\Rightarrow$ SNC $\Rightarrow$ GNC $\Rightarrow$ NC.
- GNC holds for all Artin algebras if and only if so does ARC for all Artin algebras.

Thus, FDC holds for all algebras implies that ARC holds for all algebras. However, the implication is unknown for individual algebras. For example, the finitistic dimension conjecture is known to be true for self-injective algebras, but ARC is still open for these algebras [20, Theorem 3.4].

ARC is known to be true for several special classes of algebras. For example, algebras of finite representation type, torsionless-finite algebras, symmetric biserial algebras, algebras with radical square zero, local algebras with radical cube zero [3,38,39]. ARC also holds for an algebra satisfying the (**Fg**) condition [16]. These algebras include group algebras of finite groups [8], finite group schemes [17], commutative complete intersections [4], quantum complete intersections where $q_{i,j}$ are roots of unity [27], and so on.

This article is devoted to studying ARC in the context of singularity categories of algebras. The singularity category $\mathscr{D}_{\mathsf{sg}}(A)$ of an algebra A is defined as the Verdier quotient

$$\mathscr{D}_{\mathsf{sg}}(A) = \mathscr{D}^{\mathrm{b}}(A\text{-}\mathsf{mod})/\mathscr{K}^{\mathrm{b}}(A\text{-}\mathsf{proj})$$

of the bounded derived category $\mathscr{D}^{\mathrm{b}}(A\text{-}\mathsf{mod})$ by the subcategory of bounded complexes of projective modules [7]. Two algebras are called *singularly equivalent* if their singularity categories are equivalent as triangulated categories.

It is well-known that there is a full embedding

$$\underline{{}^{\perp}A} \hookrightarrow \mathscr{D}_{\mathsf{sg}}(A)$$

sending a module to the corresponding stalk complex concentrated in degree zero, where

$${}^{\perp}A := \{X \in A\text{-}\mathsf{mod} \,|\, \mathrm{Ext}^i_A(X, A) = 0 \text{ for all } i > 0\}$$

and $\underline{{}^{\perp}A}$ is the additive quotient category of ${}^{\perp}A$ modulo projective modules. The embedding induces an isomorphism

$$\mathrm{Ext}^i_A(X, Y) \cong \mathrm{Hom}_{\mathscr{D}_{\mathsf{sg}}(A)}(X, Y[i])$$

for each $X, Y \in {}^{\perp}A$ and for each $i > 0$. Here, we write $Y[i]$ for the object obtained from Y by applying the shift functor i times.

ARC holds for A precisely means that $\underline{{}^{\perp}A}$ has no nonzero objects X with $\mathrm{Ext}^i_A(X, X) = 0$ for all positive i. Recall that an object T in a triangulated category $\mathcal{T}$ is *presilting* if $\mathrm{Hom}_{\mathcal{T}}(T, T[i]) = 0$ for all $i > 0$. As a consequence, if $\mathscr{D}_{\mathsf{sg}}(A)$ has no nonzero presilting objects, then ARC holds for A. Therefore, it is natural to conjecture: *The singularity category of any algebra contains no nonzero presilting objects*. We call this singular presilting conjecture (SPC). Observe that SPC implies ARC, and the converse is also true if A is a Gorenstein algebra — the above embedding is an equivalence in this case. Hence SPC might be used as a tool to study ARC. Obviously, SPC is invariant under singular equivalences. This implies that ARC is preserved under singular equivalences between Gorenstein algebras. One can ask a more general question.

Question: *Do singular equivalences preserve ARC?*

If two algebras are derived equivalent (they are certainly also singularly equivalent), then the answer to the above question is yes. This was proved by Wei in [37] (see [30,15] for derived invariance of a generalized version of ARC). In this paper, we shall consider singular equivalences induced by adjoint pairs and show that many singular equivalences do preserve ARC.

Our first result is the following, which is listed in Theorem 3.6.

Theorem 1.1. *Let A, B be two algebras. Suppose that there are triangle functors*

$$\begin{tikzcd}[column sep=huge]
\mathscr{D}(B\text{-}\mathsf{Mod}) \arrow[r, bend left=50, "L"] \arrow[r, "G"] \arrow[r, bend right=50, "K"'] & \mathscr{D}(A\text{-}\mathsf{Mod}) \arrow[l, bend right=20, "F"'] \arrow[l, bend left=20, "H"]
\end{tikzcd}$$

between the unbounded derived categories of A and B such that $(L,F),(F,G),(G,H)$ and (H,K) are all adjoint pairs. Assume that G induces a singular equivalence, and that H preserves bounded complexes of projective modules. Then the ARC holds for A if and only if it holds for B.

Applying Theorem 1.1 to ladders (see Section 3), we get the following theorem (see Theorem 3.8).

Theorem 1.2. *Let A, B and C be three algebras, and there is a ladder of height* 3

$$\begin{tikzcd}[column sep=huge]
\mathscr{D}(B\text{-}\mathsf{Mod}) \arrow[r, bend left=20, "i_*"] \arrow[r, bend right=20, "i_?"'] & \mathscr{D}(A\text{-}\mathsf{Mod}) \arrow[l, bend right=50, "i^*"'] \arrow[l, "i^!"] \arrow[l, bend left=50] \arrow[r, bend left=20, "j^*"] \arrow[r, bend right=20, "j^?"'] & \mathscr{D}(C\text{-}\mathsf{Mod}) \arrow[l, bend right=50, "j_!"'] \arrow[l, "j_*"] \arrow[l, bend left=50]
\end{tikzcd} .$$

Then ARC holds for A implies that it holds for B. Moreover, if the ladder can be completed to a ladder of height 4, then the following statements hold.

(1) *If the ARC holds for A, then it holds for B and C;*
(2) *If C (resp. B) has finite global dimension, then ARC holds for A if and only if it holds for B (resp. C).*

Theorem 1.1 can also be applied to singular equivalences induced by tensor functors. The following result is listed in Theorem 4.1.

Theorem 1.3. *Suppose that $X^\bullet$ is a B-A-bimodules complex which is perfect over B and A, and assume $Y^\bullet := \mathbf{RHom}_B(X^\bullet, B)$ is a perfect complex of A-modules. If $X^\bullet \otimes_A^{\mathbf{L}} -$ induces a singular equivalence between A and B, then ARC holds for B implies that it holds for A. If moreover $\mathbf{RHom}_A(Y^\bullet, A) \in \mathscr{K}^{\mathrm{b}}(B\text{-}\mathsf{proj})$, then ARC holds for A if and only if it holds for B.*

Our result can be applied to give several reduction methods on ARC. Let A be a lower triangular matrix, and e be an idempotent. Under certain conditions, A satisfies ARC if and only if so does eAe (see Corollary 3.9). As an immediate application, ARC is preserved under one-point (co-)extensions (see Example 5.3). Thus, for quiver algebras, ARC is invariant when we remove a sink or a source. Let AeA be a heredity ideal of A such that eA has finite injective dimension as a right A-module. We also show that ARC holds for A if and only if it holds for A/AeA (see Example 5.4).

This paper is organized as follows. We collect necessary facts in Section 2, and investigate ARC under singular equivalences induced by adjoint pairs and recollements in Sections 3 and 4. Some examples are given in the final section.

2. Preliminaries

In this section we fix our notation and recall some basic facts for later proofs.

As we mentioned at the beginning of the previous section, all algebras are finite dimensional algebras over a fixed field k. Let A be such an algebra. The opposite algebra of A is denoted by A^{op} and the enveloping algebra $A \otimes_k A^{\mathrm{op}}$ is denoted by A^e. We identify A-A-bimodules with left A^e-modules. We denote by A-Mod the category of left A-modules, by A-mod its subcategory consisting of finitely generated left A-modules, by A-Proj its subcategory consisting of projective left A-modules, and by A-proj its subcategory consisting of finitely generated projective left A-modules.

Let $\mathscr{X}$ be a full subcategory of A-mod, then we define

$$^{\perp}\mathscr{X} := \{Y \in A\text{-}\mathsf{mod} \mid \mathrm{Ext}^i_A(Y, X) = 0, \forall i > 0, X \in \mathscr{X}\}.$$

The corresponding additive quotient modulo projectives is denoted by $\underline{^{\perp}\mathscr{X}}$. An A-module M contained in $^{\perp}M$ is called *self-orthogonal.*

For an algebra A, the unbounded derived category of A-Mod is denoted by $\mathscr{D}(A\text{-}\mathsf{Mod})$. As usual, we write $\mathscr{D}^{\mathrm{b}}(A\text{-}\mathsf{Mod})$ (resp. $\mathscr{D}^{\mathrm{b}}(A\text{-}\mathsf{mod})$) for the full subcategory consisting of bounded complexes of left A-modules (resp. finitely generated left A-modules), and write $\mathscr{K}^{\mathrm{b}}(A\text{-}\mathsf{Proj})$ (resp. $\mathscr{K}^{\mathrm{b}}(A\text{-}\mathsf{proj})$) for the full subcategory consisting of bounded complexes of projective modules (resp. finitely generated projective modules).

It is well-known that the above mentioned unbounded and bounded derived categories are all triangulated categories. We refer to Happel's book [19] for basic results on triangulated categories.

The following result is well-known. Here, we give a proof for readers' convenience; compare [33, Theorem 2.1] and [2, Proposition 5.1.7].

Lemma 2.1. *Let $\mathcal{A}$ be an abelian category and $\mathcal{P}$ be the full subcategory of projective objects. Then the canonical functor*

$$\iota : \underline{^{\perp}\mathcal{P}} \longrightarrow \mathscr{D}^{\mathrm{b}}(\mathcal{A})/\mathscr{K}^{\mathrm{b}}(\mathcal{P})$$

is fully faithful.

Proof. A morphism $X^\bullet \longrightarrow Y^\bullet$ in $\mathscr{D}^{\mathrm{b}}(\mathcal{A})/\mathscr{K}^{\mathrm{b}}(\mathcal{P})$ is denoted by a fraction $as^{-1} : X^\bullet \xleftarrow{s} Z^\bullet \xrightarrow{a} Y^\bullet$, where a and s are morphisms in $\mathscr{D}^{\mathrm{b}}(\mathcal{A})$, and if $Z^\bullet \xrightarrow{s} X^\bullet \longrightarrow U^\bullet \longrightarrow Z^\bullet[1]$ is a triangle in $\mathscr{D}^{\mathrm{b}}(\mathcal{A})$, then $U^\bullet \in \mathscr{K}^{\mathrm{b}}(\mathcal{P})$. A morphism s' in $\mathscr{D}^{\mathrm{b}}(\mathcal{A})$ with this property will be denoted by $\overset{s'}{\Longrightarrow}$.

First, we show that ι is a full functor. Let $f : X \longrightarrow Y$ be a morphism in ${}^\perp\mathcal{P}$. Then $\iota(f)$ is the morphism $X \overset{1_X}{\Longleftarrow} X \xrightarrow{f} Y$. We need to show that each morphism from X to Y in $\mathscr{D}^{\mathrm{b}}(\mathcal{A})/\mathscr{K}^{\mathrm{b}}(\mathcal{P})$ is of this form. Let $X \overset{s}{\Longleftarrow} U^\bullet \xrightarrow{a} Y$ be a morphism in $\mathscr{D}^{\mathrm{b}}(\mathcal{A})/\mathscr{K}^{\mathrm{b}}(\mathcal{P})$. By definition, there is a triangle $U^\bullet \xrightarrow{s} X \xrightarrow{g} P^\bullet \longrightarrow U^\bullet[1]$ in $\mathscr{D}^{\mathrm{b}}(\mathcal{A})$ with $P^\bullet \in \mathscr{K}^{\mathrm{b}}(\mathcal{P})$. Consider the triangle in $\mathscr{D}^{\mathrm{b}}(\mathcal{A})$

$$\sigma_{\geq 0}P^\bullet \xrightarrow{\alpha} P^\bullet \xrightarrow{\beta} \sigma_{<0}P^\bullet \longrightarrow (\sigma_{\geq 0}P^\bullet)[1].$$

Here σ denotes the brutal truncation. Since $P^\bullet$ belongs to $\mathscr{K}^{\mathrm{b}}(\mathcal{P})$ and $X \in {}^\perp\mathcal{P}$, by [21, Proposition 3.1] we have

$$\mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(\mathcal{A})}(X, \sigma_{<0}P^\bullet) \cong \mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A)}(X, \sigma_{<0}P^\bullet) = 0.$$

It follows that $\beta g = 0$, and therefore g factorizes through α. Hence we can form the following commutative diagram in $\mathscr{D}^{\mathrm{b}}(\mathcal{A})$ with rows being triangles.

$$\begin{tikzcd}
V^\bullet \arrow[r, Rightarrow, "h"] \arrow[d, Rightarrow, "r"] & X \arrow[r] \arrow[d, equal] & \sigma_{\geq 0}P^\bullet \arrow[r, "w"] \arrow[d, "\alpha"] & V^\bullet[1] \arrow[d, Rightarrow, "r[1]"] \\
U^\bullet \arrow[r, Rightarrow, "s"] & X \arrow[r, "g"] & P^\bullet \arrow[r] & U^\bullet[1].
\end{tikzcd}$$

Since $\mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(\mathcal{A})}((\sigma_{\geq 0}P^\bullet)[-1], Y) \cong \mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A)}((\sigma_{\geq 0}P^\bullet)[-1], Y) = 0$, the morphism $ar(w[-1]) = 0$, and hence there is some morphism $f : X \longrightarrow Y$ in $\mathscr{D}^{\mathrm{b}}(\mathcal{A})$ such that $ar = fh$. Then we have the following commutative diagram in $\mathscr{D}^{\mathrm{b}}(\mathcal{A})$

$$\begin{tikzcd}
 & V^\bullet \arrow[dl, Rightarrow, "r"'] \arrow[dr, Rightarrow, "h"] & \\
U^\bullet \arrow[d, Rightarrow, "s"'] \arrow[drr, "a"' near end] & & X \arrow[d, "f"] \arrow[dll, Rightarrow, "1_X"' near start] \\
X & & Y,
\end{tikzcd}$$

which means that the morphisms $X \overset{s}{\Longleftarrow} U^\bullet \xrightarrow{a} Y$ and $X \overset{1_X}{\Longleftarrow} X \xrightarrow{f} Y$ in $\mathscr{D}^{\mathrm{b}}(\mathcal{A})/\mathscr{K}^{\mathrm{b}}(\mathcal{P})$ are equal. Hence ι is a full functor.

Suppose that $f : X \longrightarrow Y$ is a morphism in $\mathcal{A}$ such that $\iota(f) = 0$. That is, the morphisms $X \overset{1_X}{\Longleftarrow} X \xrightarrow{0} Y$ and $X \overset{1_X}{\Longleftarrow} X \xrightarrow{f} Y$ are equal in $\mathscr{D}^{\mathrm{b}}(\mathcal{A})/\mathscr{K}^{\mathrm{b}}(\mathcal{P})$. Then there is a morphism $W^\bullet \overset{s}{\Longrightarrow} X$ such that $fs = 0$ in $\mathscr{D}^{\mathrm{b}}(\mathcal{A})$. Embedding s into a triangle in $\mathscr{D}^{\mathrm{b}}(\mathcal{A})$, we see that f factorizes in $\mathscr{D}^{\mathrm{b}}(\mathcal{A})$ through a complex in $\mathscr{K}^{\mathrm{b}}(\mathcal{P})$. Since $X \in {}^\perp\mathcal{P}$

and $\mathcal{P}$ consists of projective objects in $\mathcal{A}$, it follows that $\mathrm{Ext}^i_{\mathcal{A}}(X, P) = 0 = \mathrm{Ext}^i_{\mathcal{A}}(P, Y)$ for all $P \in \mathcal{P}$ and all $i > 0$. By [21, Corollary 3.4], f factorizes through an object in $\mathcal{P}$, that is, $f = 0$ in the stable category. Hence the functor ι is faithful. $\square$

An object X in a triangulated category is called *presilting* if $\mathrm{Hom}(X, X[n]) = 0$ for all $n > 0$. By Lemma 2.1, there is a full embedding

$$\underline{{}^{\perp}A} \hookrightarrow \mathscr{D}_{\mathsf{sg}}(A)$$

which induces a natural isomorphism

$$\mathrm{Ext}^i_A(X, Y) \cong \underline{\mathrm{Hom}}_A(\Omega^i(X), Y) \cong \mathrm{Hom}_{\mathscr{D}_{\mathsf{sg}}(A)}(X, Y[i])$$

for each $i > 0$; see [34, Lemma 5.2] for example. Thus, every self-orthogonal object in $\underline{{}^{\perp}A}$ is a presilting object in $\mathscr{D}_{\mathsf{sg}}(A)$.

Let $X^\bullet$ be a complex of (finitely generated) A-modules. The i-th homology of $X^\bullet$ is denoted by $H^i(X^\bullet)$. For a right A-module M and a left A-module N, denote by $M \otimes_A X^\bullet$ and $\mathrm{Hom}_A(X^\bullet, N)$ the complexes

$$\cdots \to M \otimes_A X^i \xrightarrow{1 \otimes d^i} M \otimes_A X^{i+1} \to \cdots$$

and

$$\cdots \to \mathrm{Hom}_A(X^{i+1}, N) \xrightarrow{\mathrm{Hom}_A(d^i, N)} \mathrm{Hom}_A(X^i, N) \to \cdots$$

respectively. Note that the i-th term of $\mathrm{Hom}_A(X^\bullet, N)$ is $\mathrm{Hom}_A(X^{-i}, N)$ for all $i \in \mathbb{Z}$.

The following will be useful in our later discussion.

Lemma 2.2 *([25, Theorem 4.1 and 5.1] and [1, Lemma 2.7]). Let A and B be two algebras, and $F : \mathscr{D}(A\text{-}\mathsf{Mod}) \to \mathscr{D}(B\text{-}\mathsf{Mod})$ be a triangle functor with a right adjoint G. Consider the following conditions*

(1) *F preserves $\mathscr{K}^{\mathrm{b}}(\mathsf{proj})$;*
(2) *G preserves coproducts;*
(3) *G admits a right adjoint;*
(4) *G preserves $\mathscr{D}^{\mathrm{b}}(\mathsf{mod})$;*
(5) *G preserves $\mathscr{D}^{\mathrm{b}}(\mathsf{Mod})$.*

Then we have (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Rightarrow$ (5)*.*

3. Singular equivalences induced by adjoint tuples

In this section, we consider ARC and singular equivalences induced by adjoint pairs. First, let us recall from [21] the definition of non-negative functors.

Definition 3.1 *([21, Definition 4.1]).* Let A and B be two algebras. A triangle functor

$$G : \mathscr{D}^{\mathrm{b}}(B\text{-}\mathsf{Mod}) \to \mathscr{D}^{\mathrm{b}}(A\text{-}\mathsf{Mod})$$

is called *non-negative* if G satisfies the following two conditions:

(1) $G(X)$ is isomorphic to a complex with zero homology in all negative degrees for all $X \in B\text{-}\mathsf{Mod}$;
(2) $G(P)$ is isomorphic to a complex in $\mathscr{K}^{\mathrm{b}}(A\text{-}\mathsf{Proj})$ with zero terms in all negative degrees for all $P \in B\text{-}\mathsf{Proj}$.

The following proposition taken from [21] on non-negative functors will be crucial for our discussion.

Proposition 3.2 *([21, Proposition 4.8 and 5.2]). Let $G : \mathscr{D}^{\mathrm{b}}(B\text{-}\mathsf{Mod}) \to \mathscr{D}^{\mathrm{b}}(A\text{-}\mathsf{Mod})$ be a non-negative triangle functor admitting a right adjoint H which preserves $\mathscr{K}^{\mathrm{b}}(\mathrm{Proj})$. If G restricts to $\mathscr{D}^{\mathrm{b}}(\mathsf{mod})$ and $\mathscr{K}^{\mathrm{b}}(\mathrm{proj})$, then there is a commutative diagram*

$$\begin{tikzcd} \underline{{}^{\perp}B} \arrow[r, "\overline{G}"] \arrow[d, hook, "\iota_B"] & \underline{{}^{\perp}A} \arrow[d, hook, "\iota_A"] \\ \mathscr{D}_{\mathsf{sg}}(B) \arrow[r, "\tilde{G}"] & \mathscr{D}_{\mathsf{sg}}(A), \end{tikzcd} \tag{3.1}$$

where ι_B and ι_A are natural embeddings.

In certain cases, the above commutative diagram indicates the relationship between the validity of ARC for A and B, as is shown in the following lemma.

Lemma 3.3. *Assume that we have the above commutative diagram (3.1). If $\tilde{G}$ is fully faithful, then ARC holds for A implies that it holds for B.*

Proof. Let X be a self-orthogonal object in $\underline{{}^{\perp}B}$. Since the embedding ι_B takes self-orthogonal objects to presilting objects, the object $\iota_B(X)$ and thus $\tilde{G}\iota_B(X)$ is presilting. By the commutative diagram (3.1), one has $\iota_A\bar{G}(X) = \tilde{G}\iota_B(X)$. It follows that $\bar{G}(X)$ is self-orthogonal in $\underline{{}^{\perp}A}$. By assumption ARC holds for A. This implies that $\bar{G}(X)$ is isomorphic to the zero object in $\underline{{}^{\perp}A}$. Since $\tilde{G}$ is fully faithful, by the commutative diagram (3.1), one can see that $\bar{G}$ is fully faithful. Hence X must be isomorphic to the zero object in $\underline{{}^{\perp}B}$.

Altogether, we have shown that $\underline{{}^{\perp}B}$ does not contain any nonzero self-orthogonal objects, that is, ARC holds for B. $\square$

Lemma 3.3 will serve as our main idea to study the relationship between singular equivalences and ARC. However, at this stage, we don't know how to get non-negative functors that induce singular equivalences or fully faithful functors between singularity categories.

If G is a derived equivalence, then $G[i]$ is non-negative for some integer i. In general, the following lemma shows that adjoint pairs of triangle functors between derived categories may give rise to non-negative functors.

Lemma 3.4. *Let*

$$\begin{tikzcd} \mathscr{D}(B\text{-}\mathsf{Mod}) \arrow[r, bend right, "G"'] & \mathscr{D}(A\text{-}\mathsf{Mod}) \arrow[l, bend right, "F"'] \end{tikzcd}$$

be an adjoint pair with both F and G preserving $\mathscr{K}^{\mathrm{b}}(\mathsf{proj})$. Then, up to shifts, G restricts to a non-negative functor from $\mathscr{D}^{\mathrm{b}}(B\text{-}\mathsf{Mod})$ to $\mathscr{D}^{\mathrm{b}}(A\text{-}\mathsf{Mod})$.

Proof. Since F preserves $\mathscr{K}^{\mathrm{b}}(\mathsf{proj})$, it follows from Lemma 2.2 that G preserves $\mathscr{D}^{\mathrm{b}}(\mathsf{Mod})$ and coproducts.

By assumption both $G(B)$ and $F(A)$ are bounded complexes of finitely generated projectives. Assume that $F(A)$ and $G(B)$ are of the following form:

$$G(B): \quad \cdots \to 0 \to P^{-m} \to \cdots \to P^0 \to P^1 \to \cdots \to P^n \to 0 \to \cdots,$$
$$F(A): \quad \cdots \to 0 \to Q^{-r} \to \cdots \to Q^0 \to Q^1 \to \cdots \to Q^s \to 0 \to \cdots.$$

Set $t = \max\{m, s\}$, $\hat{F} = F[t]$ and $\hat{G} = G[-t]$. Then $\hat{G}(B) = G(B)[-t]$ is a complex in $\mathscr{K}^{\mathrm{b}}(A\text{-}\mathsf{proj})$ with zero terms in all negative degrees, and $\hat{F}(A) = F(A)[t]$ is a complex in $\mathscr{K}^{\mathrm{b}}(B\text{-}\mathsf{proj})$ with zero terms in all positive degrees. Since $\hat{G}$ preserves coproducts, the complex $\hat{G}(P)$ is isomorphic to a complex in $\mathscr{K}^{\mathrm{b}}(A\text{-}\mathsf{Proj})$ with zero terms in all negative degrees for all $P \in B\text{-}\mathsf{Proj}$.

For any $X \in B\text{-}\mathsf{Mod}$ and any integer i, we have isomorphisms

$$\begin{aligned} H^i(\hat{G}(X)) &\cong \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A\text{-}\mathsf{Mod})}(A, \hat{G}(X)[i]) \\ &\cong \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(B\text{-}\mathsf{Mod})}(\hat{F}(A), X[i]) \\ &\cong \mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(B\text{-}\mathsf{Mod})}(\hat{F}(A), X[i]), \end{aligned}$$

where the second isomorphism follows from the adjointness of $\hat{F}$ and $\hat{G}$. Therefore, $H^i(\hat{G}(X)) = 0$ for any $i < 0$, that is, $\hat{G}(X)$ has no homology in negative degrees. This proves that $\hat{G}$ is non-negative. $\square$

Corollary 3.5. *Suppose that (F, G, H) in the following diagram is an adjoint triple of triangulated functors.*

$$\begin{tikzcd}[column sep=huge] \mathscr{D}(B\text{-}\mathsf{Mod}) \arrow[r, "G"] & \mathscr{D}(A\text{-}\mathsf{Mod}) \arrow[l, bend right, "F"'] \arrow[l, bend left, "H"] \end{tikzcd}$$

Assume that G preserves $\mathscr{K}^{\mathrm{b}}(\mathsf{proj})$ and H preserves $\mathscr{K}^{\mathrm{b}}(\mathsf{Proj})$. Then we have the commutative diagram (3.1). Moreover, if G induces a fully faithful functor between the singularity categories of B and A, then ARC holds for A implies that it holds for B.

Proof. Since G has a right adjoint H, by Lemma 2.2, we have that F preserves $\mathscr{K}^{\mathrm{b}}(\mathsf{proj})$ and G preserves $\mathscr{D}^{\mathrm{b}}(\mathsf{mod})$. It follows from Lemma 3.4 that, up to shifts, G restricts to a non-negative functor from $\mathscr{D}^{\mathrm{b}}(B\text{-}\mathsf{Mod})$ to $\mathscr{D}^{\mathrm{b}}(A\text{-}\mathsf{Mod})$. Since G preserves $\mathscr{D}^{\mathrm{b}}(\mathsf{mod})$ and $\mathscr{K}^{\mathrm{b}}(\mathsf{proj})$, one gets the commutative diagram (3.1) by Proposition 3.2. Moreover, the statement follows from Lemma 3.3. $\square$

A sequence of functors $(F_1, F_2, \ldots, F_r)$ between two categories is called an *adjoint tuple* if (F_i, F_{i+1}) is an adjoint pair for all $i = 1, \ldots, r-1$.

Theorem 3.6. *Let A and B be two algebras. Suppose that the sequence $(F_1, F_2, F_3, F_4, F_5)$ of triangle functors in the following diagram is an adjoint tuple.*

$$\begin{tikzcd}[column sep=huge] \mathscr{D}(B\text{-}\mathsf{Mod}) \arrow[r, bend left=50, "F_1"] \arrow[r, "F_3"] \arrow[r, bend right=50, "F_5"'] & \mathscr{D}(A\text{-}\mathsf{Mod}) \arrow[l, bend right, "F_2"'] \arrow[l, bend left, "F_4"] \end{tikzcd}$$

Assume that F_3 induces a singular equivalence between B and A. If F_4 preserves $\mathscr{K}^{\mathrm{b}}(\mathsf{Proj})$, then ARC holds for A if and only if it holds for B.

Proof. By Lemma 2.2, the functors F_1, F_2 and F_3 preserve $\mathscr{K}^{\mathrm{b}}(\mathsf{proj})$. Now consider the adjoint triple (F_2, F_3, F_4). Since F_4 preserves $\mathscr{K}^{\mathrm{b}}(\mathsf{Proj})$ and F_3 induces a singular equivalence, by Corollary 3.5, ARC holds for A implies that it holds for B.

Conversely, consider the adjoint triple (F_1, F_2, F_3). By Lemma 2.2, the functors F_2 and F_3 preserve both $\mathscr{K}^{\mathrm{b}}(\mathsf{proj})$ and $\mathscr{D}^{\mathrm{b}}(\mathsf{mod})$. Thus, F_2 and F_3 induce an adjoint pair between the singularity categories of A and B. Since F_3 induces a singular equivalence, so does F_2. Moreover F_3 also preserves $\mathscr{K}^{\mathrm{b}}(\mathsf{Proj})$ since it preserves both $\mathscr{K}^{\mathrm{b}}(\mathsf{proj})$ and coproducts. Hence, applying Corollary 3.5 to the adjoint triple (F_1, F_2, F_3), we conclude that ARC holds for B implies that it holds for A. $\square$

Corollary 3.7. *Let A and B be two algebras. Suppose that the sequence $(F_1, \dots, F_6)$ of triangle functors in the following diagram is an adjoint tuple.*

$$\begin{tikzcd}[column sep=huge]
\mathscr{D}(B\text{-}\mathsf{Mod}) \arrow[r, bend left=50, "F_1"] \arrow[r, "F_3"] \arrow[r, bend right=30, "F_5"'] & \mathscr{D}(A\text{-}\mathsf{Mod}) \arrow[l, bend right=20, "F_2"'] \arrow[l, bend left=20, "F_4"] \arrow[l, bend left=60, "F_6"]
\end{tikzcd}$$

Assume that F_3 induces a singular equivalence between B and A. Then ARC holds for A if and only if it holds for B.

Proof. According to Theorem 3.6, it suffices to prove that F_4 preserves $\mathscr{K}^{\mathrm{b}}(\mathsf{Proj})$. However, this follows easily from the fact that F_4 preserves $\mathscr{K}^{\mathrm{b}}(\mathsf{proj})$ and coproducts by Lemma 2.2. $\square$

An immediate consequence of Corollary 3.7 is that derived equivalences preserve ARC, which was proved by Wei in [37]. Actually, a derived equivalence F and its quasi-inverse F^{-1} give rise to an adjoint tuple $(F, F^{-1}, F, F^{-1}, \dots)$ of arbitrary length, and F induces a singular equivalence.

Adjoint tuples typically occur in recollements and ladders.

Let $\mathcal{T}_1$, $\mathcal{T}$ and $\mathcal{T}_2$ be triangulated categories. A *recollement* [5] of $\mathcal{T}$ relative to $\mathcal{T}_1$ and $\mathcal{T}_2$ is a diagram

$$\begin{tikzcd}[column sep=huge]
\mathcal{T}_1 \arrow[r, "i_*=i_!" description] & \mathcal{T} \arrow[l, bend right=30, "i^*" description] \arrow[l, bend left=30, "i^!" description] \arrow[r, "j^!=j^*" description] & \mathcal{T}_2 \arrow[l, bend right=30, "j_!" description] \arrow[l, bend left=30, "j_*" description]
\end{tikzcd}$$

of triangulated categories and triangle functors such that

(1) $(i^*, i_*, i^!), (j_!, j^!, j_*)$ are adjoint triples of triangle functors;
(2) i_*, $j_!$ and j_* are full embeddings;
(3) $j^! i_* = 0$ (and thus also $i^! j_* = 0$ and $i^* j_! = 0$);
(4) for each $X \in \mathcal{T}$, there are triangles

$$\begin{array}{c} j_! j^! X \to X \to i_* i^* X \to \\ i_! i^! X \to X \to j_* j^* X \to \end{array},$$

where the maps are given by adjunctions.

A *ladder* [1] is a finite or infinite diagram of triangulated categories and triangle functors

$$\begin{tikzcd}[column sep=huge] \vdots & & \vdots \\ \mathcal{T}_1 \arrow[r, bend left=20] \arrow[r, bend right=20] & \mathcal{T} \arrow[l, bend right=50] \arrow[l] \arrow[l, bend left=50] \arrow[r, bend left=20] \arrow[r, bend right=20] & \mathcal{T}_2 \arrow[l, bend right=50] \arrow[l] \arrow[l, bend left=50] \\ \vdots & & \vdots \end{tikzcd}$$

such that any three consecutive rows form a recollement. The *height* of a ladder is the number of recollements contained in it (counted with multiplicities).

Now, we are ready to apply our discussion to ladders.

Theorem 3.8. *Suppose that A, B and C are algebras, and there is a ladder of height 3*

$$\begin{tikzcd}[column sep=huge] \mathscr{D}(B\text{-}\mathsf{Mod}) \arrow[r, bend left=20, "i_*" description] \arrow[r, bend right=20, "i_?" description] & \mathscr{D}(A\text{-}\mathsf{Mod}) \arrow[l, bend right=50, "i^*" description] \arrow[l, "i^!" description] \arrow[l, bend left=50] \arrow[r, bend left=20, "j^*" description] \arrow[r, bend right=20, "j^?" description] & \mathscr{D}(C\text{-}\mathsf{Mod}) \arrow[l, bend right=50, "j_!" description] \arrow[l, "j_*" description] \arrow[l, bend left=50] \end{tikzcd} .$$

Then ARC holds for A implies that it holds for B. Moreover, if the ladder can be completed to a ladder of height 4, *then the following statements hold.*

(1). *If the ARC holds for A, then it holds for B and C;*
(2). *i_* (resp. j^*) induces a singular equivalence if and only if C (resp. B) has finite global dimension, and in this case, the ARC holds for A if and only if it holds for B (resp. C).*

Proof. It follows from Lemma 2.2 that i^*, i_* and $i^!$ preserve $\mathscr{K}^{\mathrm{b}}(\mathsf{proj})$, and $i^!$ preserves coproducts. Therefore, $i^!$ preserves $\mathscr{K}^{\mathrm{b}}(\mathsf{Proj})$. Moreover, i_* is a fully faithful functor which preserves $\mathscr{K}^{\mathrm{b}}(\mathsf{proj})$ and $\mathscr{D}^{\mathrm{b}}(\mathsf{mod})$, and then the induced functor $\tilde{i}_* : \mathscr{D}_{\mathsf{sg}}(B) \to \mathscr{D}_{\mathsf{sg}}(A)$ is also fully faithful, see [29, Lemma 1.1] or [9, Lemma 2.2]. Applying Corollary 3.5 to the adjoint triple $(i^*, i_*, i^!)$, we get that the ARC holds for B provided it holds for A.

Assume the ladder can be completed to a ladder of height 4. Without loss of generality, we may assume it extends one step downwards, and the upward case can be proved similarly. Consider the following ladder of height 4

$$\begin{tikzcd}[column sep=huge] \mathscr{D}(B\text{-}\mathsf{Mod}) \arrow[r, bend left=20, "i_*" description] \arrow[r, bend right=20, "i_?" description] \arrow[r, bend right=70] & \mathscr{D}(A\text{-}\mathsf{Mod}) \arrow[l, bend right=50, "i^*" description] \arrow[l, "i^!" description] \arrow[l, bend left=50] \arrow[r, bend left=20, "j^*" description] \arrow[r, bend right=20, "j^?" description] \arrow[r, bend right=70] & \mathscr{D}(C\text{-}\mathsf{Mod}) \arrow[l, bend right=50, "j_!" description] \arrow[l, "j_*" description] \arrow[l, bend left=50] \end{tikzcd} \tag{3.2}$$

Then, it follows from Lemma 2.2 that j_* and $j^?$ preserve $\mathscr{K}^{\mathrm{b}}(\mathsf{proj})$. Moreover, $j^?$ preserves coproducts and then it preserves $\mathscr{K}^{\mathrm{b}}(\mathsf{Proj})$. On the other hand, j_* induces a fully faithful functor between the corresponding singularity categories. Now, applying Corollary 3.5 to the adjoint triple $(j^*, j_*, j^?)$, we get that the ARC holds for C if it holds for A.

By [22, Proposition 2.5], the functors $i_*, i^!, i_?, j^*, j_*$ and $j^?$ induce a recollement between the corresponding singularity categories. Therefore, i_* induces a singular equivalence if and only if $\mathscr{D}_{\mathsf{sg}}(C) = 0$, and this occurs precisely when C has finite global dimension. In this case, $i^!$ also induces a singular equivalence. Applying Corollary 3.7 to the left part of the ladder (3.2), we obtain that the ARC holds for A if and only if it holds for B. Similarly, j^* induces a singular equivalence if and only if B has finite global dimension, and in this case, the ARC holds for A if and only if it holds for C. $\square$

Theorem 3.8 can be applied to triangle matrix algebras.

Corollary 3.9. *Let* $A = \begin{pmatrix} B & 0 \\ {}_CM_B & C \end{pmatrix}$ *be a triangular matrix algebra, where* B, C *are algebras and* M *a finitely generated* C*-*B*-bimodules. Then following statements hold.*

(1). *If* $\operatorname{gl.dim} B < \infty$ *and* $\operatorname{proj.dim}_C M < \infty$*, then the ARC holds for* A *if and only if it holds for* C*;*

(2). *If* $\operatorname{gl.dim} C < \infty$ *and* $\operatorname{proj.dim} M_B < \infty$*, then the ARC holds for* A *if and only if it holds for* B*.*

Proof. Let $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. It follows from [1, Example 3.4] that there is a ladder of height 2

$$\begin{tikzcd}
\mathscr{D}(C\text{-}\mathsf{Mod}) \arrow[r, shift left, "Ae_2\otimes^{\mathbf{L}}_C-"] \arrow[r, bend right=40] & \mathscr{D}(A\text{-}\mathsf{Mod}) \arrow[l, shift left, "e_2A\otimes^{\mathbf{L}}_A-"] \arrow[l, bend right=40] \arrow[r, shift left] \arrow[r, bend right=40] & \mathscr{D}(B\text{-}\mathsf{Mod}). \arrow[l, shift left] \arrow[l, bend right=40, "Ae_1\otimes^{\mathbf{L}}_B-"']
\end{tikzcd} \tag{3.3}$$

If $\operatorname{gl.dim} B < \infty$ and $\operatorname{proj.dim}_C M < \infty$, then $\operatorname{proj.dim} M_B < \infty$ and by [1, Example 3.4], the ladder (3.3) can be extended to a ladder of height 4. Applying Theorem 3.8 to this ladder, we have that the ARC holds for A if and only if it holds for C. The second case can be proved similarly. $\square$

Recall that an A-module M is *Gorenstein projective* if there are short exact sequences

$$0 \longrightarrow X_i \longrightarrow P_i \longrightarrow X_{i+1} \longrightarrow 0, i \in \mathbb{Z}$$

in A-mod with $X_i \in {}^{\perp}A$ and P_i projective for all i such that $M = X_0$. Particularly, all Gorenstein projective modules are contained in ${}^{\perp}A$. Denote by A-Gproj the full subcategory of A-mod consisting of all Gorenstein projective modules. The stable category of A-Gproj is a triangulated category and is contained in $\underline{{}^{\perp}A}$. Thus, one gets full embeddings

$$\underline{A\text{-Gproj}} \hookrightarrow \underline{{}^{\perp}A} \hookrightarrow \mathscr{D}_{\mathsf{sg}}(A).$$

If ARC holds for A, that is, $\underline{{}^{\perp}A}$ does not contain any nonzero self-orthogonal objects, then there is no nonzero self-orthogonal objects in $\underline{A\text{-Gproj}}$. It is then natural to conjecture:

Gorenstein projective conjecture (**GPC** for short): A finitely generated Gorenstein projective module M over A is projective if $\mathrm{Ext}^i_A(M, M) = 0$, for any $i \geq 1$.

This conjecture was stated by Luo and Huang in [23]. Let $G : \mathscr{D}^{\mathrm{b}}(B\text{-mod}) \to \mathscr{D}^{\mathrm{b}}(A\text{-mod})$ be a non-negative triangle functor (a small module version of Definition 3.1) admitting a right adjoint H which preserves $\mathscr{K}^{\mathrm{b}}(\mathrm{proj})$. Due to [21, Proposition 5.3], we have a commutative diagram

$$\begin{tikzcd} \underline{B\text{-Gproj}} \arrow[r, "\overline{G}"] \arrow[d, hook, "\iota_B"] & \underline{A\text{-Gproj}} \arrow[d, hook, "\iota_A"] \\ \mathscr{D}_{\mathsf{sg}}(B) \arrow[r, "\tilde{G}"] & \mathscr{D}_{\mathsf{sg}}(A), \end{tikzcd}$$

analogous to Proposition 3.2. Therefore, if we replace ARC by GPC, all results in this section (e.g., Theorem 3.6 and Theorem 3.8) still hold and the proofs are almost identical to the case of ARC.

4. Singular equivalences induced by tensor functors

Let A and B be two algebras, and $X^{\bullet}$ be a complex of B-A-bimodules. Then the tensor product functor $F = X^{\bullet} \otimes^{\mathbf{L}}_A - : \mathscr{D}(A\text{-Mod}) \to \mathscr{D}(B\text{-Mod})$ has a right adjoint $\mathbf{RHom}_B(X^{\bullet}, -)$. It is natural to ask when F induces an equivalence between the singularity categories, and under which conditions this singular equivalence preserves ARC.

It does happen often that F induces a singular equivalence. Typical examples are singular equivalences of Morita type (with level) when $X^{\bullet}$ is a module ([41, Theorem 3.1], [36, Definition 2.1]). See [10, Proposition 4.8] for recent progress in this direction. In the general case, it was proved recently by Dalezios [14, Theorem 3.6] that F induces a singular equivalence if $Y^{\bullet} := \mathbf{RHom}_B(X^{\bullet}, B)$ is a perfect complex of A-modules, $\mathbf{RHom}_B(X^{\bullet}, X^{\bullet}) \cong A$ in $\mathscr{D}_{\mathsf{sg}}(A^e)$, and $X^{\bullet} \otimes^{\mathbf{L}}_A \mathbf{RHom}_B(X^{\bullet}, B) \cong B$ in $\mathscr{D}_{\mathsf{sg}}(B^e)$.

The main result of this section is the following theorem.

Theorem 4.1. *Suppose that $X^{\bullet}$ is a B-A-bimodules complex which is perfect over B and A, and assume $Y^{\bullet} := \mathbf{RHom}_B(X^{\bullet}, B)$ is a perfect complex of A-modules. If ${}_BX^{\bullet} \otimes^{L}_A - : \mathscr{D}(A\text{-Mod}) \to \mathscr{D}(B\text{-Mod})$ induces a singular equivalence, then ARC holds for B implies* *that it holds for* A*. If moreover* $\mathbf{RHom}_A(Y^\bullet, A) \in \mathscr{K}^{\mathrm{b}}(B\text{-}\mathsf{proj})$*, then ARC holds for* A *if and only if it holds for* B*.*

Proof. Note that there is an adjoint tuple

$$\begin{tikzcd}[column sep=huge]
\mathscr{D}(B\text{-}\mathsf{Mod}) \arrow[r, bend left=50, "L"] \arrow[r, shift right, "Y^\bullet\otimes^{\mathbf{L}}_B -"'] & \mathscr{D}(A\text{-}\mathsf{Mod}) \arrow[l, shift right, "X^\bullet\otimes^{\mathbf{L}}_A -"'] \arrow[l, bend left=50, "\mathbf{RHom}_A(Y^\bullet,-)"]
\end{tikzcd}$$

with $X^\bullet \otimes^{\mathbf{L}}_A -$ and $Y^\bullet \otimes^{\mathbf{L}}_B -$ preserving $\mathscr{K}^{\mathrm{b}}(\mathsf{proj})$. Here, we observe that

$$Y^\bullet \otimes^{\mathbf{L}}_B - = \mathbf{RHom}_B(X^\bullet, B) \otimes^{\mathbf{L}}_B - \cong \mathbf{RHom}_B(X^\bullet, -),$$

since ${}_BX^\bullet$ is perfect, and thus $(X^\bullet \otimes^{\mathbf{L}}_A -, Y^\bullet \otimes^{\mathbf{L}}_B -)$ is an adjoint pair. The existence of L follows from the assumption that $X^\bullet_A$ is a perfect complex; see [1, Lemma 2.8]. Clearly, $Y^\bullet \otimes^{\mathbf{L}}_B -$ preserves coproducts and thus $Y^\bullet \otimes^{\mathbf{L}}_B -$ preserves $\mathscr{K}^{\mathrm{b}}(\mathsf{Proj})$. Now notice that since $X^\bullet \otimes^{\mathbf{L}}_A -$ induces a singular equivalence between B and A, by Corollary 3.5, ARC holds for B implies that it holds for A.

Since $Y^\bullet \otimes^{\mathbf{L}}_B -$ preserves $\mathscr{K}^{\mathrm{b}}(\mathsf{proj})$, it follows from Lemma 2.2 that $\mathbf{RHom}_A(Y^\bullet, -)$ preserves coproducts, and it admits a right adjoint K. Hence, the condition $\mathbf{RHom}_A(Y^\bullet, A) \in \mathscr{K}^{\mathrm{b}}(B\text{-}\mathsf{proj})$ implies that $\mathbf{RHom}_A(Y^\bullet, -)$ preserves $\mathscr{K}^{\mathrm{b}}(\mathsf{Proj})$. On the other hand, $(X^\bullet \otimes^{\mathbf{L}}_A -, Y^\bullet \otimes^{\mathbf{L}}_B -)$ gives an adjoint pair between the singularity categories. Since $X^\bullet \otimes^{\mathbf{L}}_A -$ induces a singular equivalence, so does $Y^\bullet \otimes^{\mathbf{L}}_B -$. Therefore, there is an adjoint tuple

$$\begin{tikzcd}[column sep=huge]
\mathscr{D}(B\text{-}\mathsf{Mod}) \arrow[r, bend left=60, "L"] \arrow[r, "Y^\bullet\otimes^{\mathbf{L}}_B -"] \arrow[r, bend right=60, "K"'] & \mathscr{D}(A\text{-}\mathsf{Mod}) \arrow[l, bend right=20, "X^\bullet\otimes^{\mathbf{L}}_A -"'] \arrow[l, bend left=20, "\mathbf{RHom}_A(Y^\bullet,-)"]
\end{tikzcd}$$

with $\mathbf{RHom}_A(Y^\bullet, -)$ preserving $\mathscr{K}^{\mathrm{b}}(\mathsf{Proj})$ and $Y^\bullet \otimes^{\mathbf{L}}_B -$ inducing a singular equivalence between B and A. Then we have done by Theorem 3.6. $\square$

Now we focus on singular equivalences from change of rings, which was studied in [28, 14]. Let $f : A \to B$ be a morphism of algebras with $\operatorname{proj.dim}_A B < \infty$ and $\operatorname{proj.dim} B_A < \infty$. Recall from [28, Lemma 3.2] that there is an adjoint pair

$$\mathscr{D}(B\text{-}\mathsf{Mod}) \underset{{}_AB\otimes^{\mathbf{L}}_B-}{\overset{{}_BB\otimes^{\mathbf{L}}_A-}{\rightleftarrows}} \mathscr{D}(A\text{-}\mathsf{Mod}). \tag{4.1}$$

If $\mathrm{Cone}(f) \in \mathscr{K}^{\mathrm{b}}(A^e\text{-}\mathsf{proj})$ (that is, $B \cong A$ in $\mathscr{D}_{\mathsf{sg}}(A^e)$) and ${}_BB \otimes^{\mathbf{L}}_A B_B \xrightarrow{\sim} {}_BB_B$ in $\mathscr{D}_{\mathsf{sg}}(B^e)$, then ${}_BB\otimes^{\mathbf{L}}_A-$ and ${}_AB\otimes^{\mathbf{L}}_B-$ induce mutual equivalences between the singularity categories; see [14, Corollary 3.7].

In particular, $f : A \to B$ is a *homological epimorphism* if the induced functor

$$f_* = {}_AB \otimes^{\mathbf{L}}_B - : \mathscr{D}(B\text{-}\mathsf{Mod}) \to \mathscr{D}(A\text{-}\mathsf{Mod})$$

is a full embedding, or equivalently, there is an isomorphism ${}_BB \otimes^{\mathbf{L}}_A B_B \xrightarrow{\sim} {}_BB_B$ in $\mathscr{D}(B^e\text{-}\mathsf{Mod})$; see [18]. In this case, ${}_AB \otimes^{\mathbf{L}}_B -$ induces a fully faithful functor between the corresponding singularity categories.

Corollary 4.2. *Let $f : A \to B$ be a morphism of algebras with* $\mathrm{proj.dim}_A B < \infty$ *and* $\mathrm{proj.dim}\, B_A < \infty$*. Then the following hold.*

(1). *If $\mathrm{Cone}(f) \in \mathscr{K}^{\mathrm{b}}(A^e\text{-}\mathsf{proj})$, then ARC holds for B implies that it holds for A;*
(2). *If moreover f is a homological epimorphism and $\mathbf{RHom}_A(B, A) \in \mathscr{K}^{\mathrm{b}}(B\text{-}\mathsf{proj})$, then the ARC holds for A if and only if it holds for B.*

Proof. (1) Since $\mathrm{proj.dim}\, B_A < \infty$, it follows from [1, Lemma 2.8] that the functor ${}_BB \otimes^{\mathbf{L}}_A -$ in diagram (4.1) has a left adjoint. Moreover, $\mathrm{proj.dim}_A B < \infty$ implies that ${}_AB \otimes^{\mathbf{L}}_B -$ preserves $\mathscr{K}^{\mathrm{b}}(\mathsf{Proj})$, and $\mathrm{Cone}(f) \in \mathscr{K}^{\mathrm{b}}(A^e\text{-}\mathsf{proj})$ yields that ${}_BB \otimes^{\mathbf{L}}_A -$ induces a fully faithful functor between the corresponding singularity categories, see [28, Proposition 3.7]. Now the statement (1) follows immediately from Corollary 3.5.

(2) Since $\mathrm{Cone}(f) \in \mathscr{K}^{\mathrm{b}}(A^e\text{-}\mathsf{proj})$ and f is a homological epimorphism, we infer that ${}_BB \otimes^{\mathbf{L}}_A -$ and ${}_AB \otimes^{\mathbf{L}}_B -$ induce mutual equivalences between the singularity categories. Now we finish our proof by Theorem 4.1. □

A special case of algebra homomorphism is the canonical map from an algebra A to its quotient A/I for some ideal I. Applying Corollary 4.2, we get the following results:

(1). If I has finite projective dimension as an A-A-bimodule, then ARC holds for A/I implies that it holds for A. Indeed, $\mathrm{proj.dim}_{A^e} I < \infty$ yields that $\mathrm{proj.dim}_A(A/I) < \infty$ and $\mathrm{proj.dim}(A/I)_A < \infty$. Moreover, it is clear that the cone of $A \to A/I$ is $I[1]$, which belongs to $\mathscr{K}^{\mathrm{b}}(A^e\text{-}\mathsf{proj})$ by assumption. So, the statement follows from Corollary 4.2 (1).
(2). A special case of (1) occurs when $I \cong M \otimes_k N$, where M and N are left and right A-modules, respectively, both have finite projective dimension. Another example is

that I is a 1-dimensional ideal which has finite projective dimension as left and as right module. We refer to [28, Corollary 3.10 and 3.11] for more explanations.

(3). In general, it is difficult to check whether the condition $\mathbf{RHom}_A(B, A) \in \mathscr{K}^{\mathrm{b}}(B\text{-}\mathsf{proj})$ in Corollary 4.2 (2) is satisfied. However, it is the case when A is a Gorenstein algebra. In fact, assume all conditions in Corollary 4.2 are satisfied, except for $\mathbf{RHom}_A(B, A) \in \mathscr{K}^{\mathrm{b}}(B\text{-}\mathsf{proj})$. Then it follows from the proof of Theorem 4.1 that there are adjoint functors

$$\begin{tikzcd} \mathscr{D}(B\text{-}\mathsf{Mod}) \arrow[r, "i_*=B\otimes^{\mathbf{L}}_B-"] \arrow[r, bend left=40, leftarrow] \arrow[r, bend right=40] & \mathscr{D}(A\text{-}\mathsf{Mod}) \arrow[l, bend right=20, "i^*=B\otimes^{\mathbf{L}}_A-"'] \arrow[l, bend left=20, "i^!"] \end{tikzcd}$$

where the functor i_* is fully faithful. Therefore, we get $i^*i_* \cong 1_B$ and $i^!i_* \cong 1_B$, and thus $B \cong i^!i_*B \in i^!(\mathscr{K}^{\mathrm{b}}(A\text{-}\mathsf{proj})) = i^!(\mathscr{K}^{\mathrm{b}}(A\text{-}\mathsf{inj}))$, which is contained in $\mathscr{K}^{\mathrm{b}}(B\text{-}\mathsf{inj})$ by [31, Lemma 1]. As a result, we obtain $\operatorname{inj.dim}_B B < \infty$, and dually, we also have $\operatorname{proj.dim}_B \operatorname{Hom}_k(B, k) < \infty$, that is, B is also a Gorenstein algebra. Therefore, $\mathbf{RHom}_A(B, A) = i^!A \in i^!(\mathscr{K}^{\mathrm{b}}(A\text{-}\mathsf{inj})) \subseteq \mathscr{K}^{\mathrm{b}}(B\text{-}\mathsf{inj}) = \mathscr{K}^{\mathrm{b}}(B\text{-}\mathsf{proj})$.

5. Examples

In this section, we illustrate our results by some examples.

Example 5.1. (Tiled orders) Let A be a finite dimensional algebra, $I_{i,j}$ be an ideal of A. Let us consider tiled triangular rings, i.e., rings of the form

$$\Delta = \begin{pmatrix} A & I_{1,2} & \cdots & I_{1,n} \\ A & A & \ddots & \vdots \\ \vdots & & \ddots & I_{n-1,n} \\ A & \cdots & \cdots & A \end{pmatrix}.$$

Here, to ensure that Δ is a ring, we need the condition $I_{i,l}I_{l,j} \subseteq I_{i,j}$ for $1 \le i < l < j \le n$; see [11, Section 1] for more details.

Proposition 5.2. *Assume that* $\operatorname{proj.dim}({}_AI_{1,i}) < \infty, \operatorname{gl.dim}(A/I_{i-1,i}) < \infty$ *for* $i = 2, \ldots, n$, *and* A *satisfies ARC. Then* Δ *satisfies ARC.*

Proof. As shown in [11, Proposition 4.14], the two algebras

$$\Delta = \begin{pmatrix} A & I_{1,2} & I_{1,3} & \cdots & I_{1,n} \\ A & A & I_{2,3} & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & I_{n-1,n} \\ A & A & A & A & A \end{pmatrix} \quad \text{and} \quad \Phi = \begin{pmatrix} A/I_{1,2} & I_{2,3}/I_{1,3} & \cdots & I_{2,n}/I_{1,n} & 0 \\ A/I_{1,2} & A/I_{1,3} & \cdots & I_{3,n}/I_{1,n} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ A/I_{1,2} & A/I_{1,3} & \cdots & I_{n-1,n}/I_{1,n} & 0 \\ A/I_{1,2} & A/I_{1,3} & \cdots & A/I_{1,n} & 0 \\ A/I_{1,2} & A/I_{1,3} & \cdots & A/I_{1,n} & A \end{pmatrix}$$

are derived equivalent. Let us denote $\Phi_1 :=$

$$\begin{pmatrix} A/I_{1,2} & I_{2,3}/I_{1,3} & \cdots & I_{2,n}/I_{1,n} \\ A/I_{1,2} & A/I_{1,3} & \cdots & I_{3,n}/I_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ A/I_{1,2} & A/I_{1,3} & \cdots & A/I_{1,n} \end{pmatrix}.$$

Since $\mathrm{proj.dim}({}_AI_{1,i}) < \infty$ for $i = 2, \dots, n$, the A-Φ_1-bimodule $(A/I_{1,2}, A/I_{1,3}, \dots, A/I_{1,n})$ has finite projective dimension as a left A-module.

Now, we want to show that Φ_1 has finite global dimension. Let e be an idempotent of Φ_1 which has 1 in the $(1,1)$-th position and zeros elsewhere. It is immediate that

$$\Phi_1 e \Phi_1 = \begin{pmatrix} A/I_{1,2} & I_{2,3}/I_{1,3} & \cdots & I_{2,n}/I_{1,n} \\ A/I_{1,2} & I_{2,3}/I_{1,3} & \cdots & I_{2,n}/I_{1,n} \\ \vdots & \vdots & & \vdots \\ A/I_{1,2} & I_{2,3}/I_{1,3} & \cdots & I_{2,n}/I_{1,n} \end{pmatrix}$$

which is projective as a right Φ_1-module. Recall that we have $I_{1,2}I_{2,i} \subseteq I_{1,i}$ for $i = 3, \dots, n$. So, $I_{2,i}/I_{1,i}$ is a left $A/I_{1,2}$-module for $i = 3, \dots, n$. It follows from $\mathrm{gl.dim}(A/I_{1,2}) < \infty$ that $I_{2,i}/I_{1,i}$ has finite projective dimension as a left $A/I_{1,2}$-module for $i = 3, \dots, n$. Hence, $\Phi_1 e \Phi_1$ has finite projective dimension as a left Φ_1-module.

It follows from $\mathrm{Tor}_i^{\Phi_1}(\Phi_1/\Phi_1 e\Phi_1, \Phi_1/\Phi_1 e\Phi_1) = 0$ for $i > 0$ that the canonical ring homomorphism $\lambda : \Phi_1 \to \Phi_1/\Phi_1 e\Phi_1$ is a homological epimorphism. Thus, we have an adjoint triple

$$\begin{tikzcd} \mathscr{D}(\text{Mod-}\Phi_1/\Phi_1 e\Phi_1) \arrow[r, "\lambda_*"] & \mathscr{D}(\text{Mod-}\Phi_1) \arrow[l, bend right, "-\otimes^{\mathbf{L}}_{\Phi_1}\Phi_1/\Phi_1 e\Phi_1"'] \arrow[l, bend left, "\mathbf{RHom}_{\Phi_1}(\Phi_1/\Phi_1 e\Phi_1,-)"] \end{tikzcd}$$

where λ_* is an embedding, and $-\otimes^{\mathbf{L}}_{\Phi_1} \Phi_1/\Phi_1 e\Phi_1$ and $\mathbf{RHom}_{\Phi_1}(\Phi_1/\Phi_1 e\Phi_1, -)$ are the derived functors of $-\otimes_{\Phi_1} \Phi_1/\Phi_1 e\Phi_1$ and $\mathrm{Hom}_{\Phi_1}(\Phi_1/\Phi_1 e\Phi_1, -)$, respectively. Note that $\Phi_1 e\Phi_1$ is projective as a right Φ_1-module, and has finite projective dimension as a left Φ_1-module. Then, the adjoint triple $(-\otimes^{\mathbf{L}}_{\Phi_1} \Phi_1/\Phi_1 e\Phi_1, \lambda_*, \mathbf{RHom}_{\Phi_1}(\Phi_1/\Phi_1 e\Phi_1, -))$ restricts to $\mathscr{D}^{\mathrm{b}}(\mathrm{Mod})$ and $\mathscr{K}^{\mathrm{b}}(\mathrm{proj})$, respectively.

By [26, Example 5.3.4], the Verdier localization of $\mathscr{D}(\text{Mod-}\Phi_1)$ via the essential image of $\mathscr{D}(\text{Mod-}\Phi_1/\Phi_1 e\Phi_1)$ under λ_* is triangle equivalent to $\mathrm{Tria}_{\mathscr{D}(\text{Mod-}\Phi_1)}(\Phi_1 e\Phi_1)$ which is the smallest full triangulated subcategory of $\mathscr{D}(\text{Mod-}\Phi_1)$ containing $\Phi_1 e\Phi_1$ and closed under small coproducts, and $\mathrm{Tria}_{\mathscr{D}(\text{Mod-}\Phi_1)}(\Phi_1 e\Phi_1)$ is triangle equivalent to the category $\mathscr{D}((\mathcal{C}_{dg}\Phi_1)(\Phi_1 e\Phi_1, \Phi_1 e\Phi_1))$ in which $(\mathcal{C}_{dg}\Phi_1)(\Phi_1 e\Phi_1, \Phi_1 e\Phi_1)$ is a dg algebra. Since $\Phi_1 e\Phi_1$ is a finitely generated projective right Φ_1-module, we have a triangle equivalence between $\mathscr{D}((\mathcal{C}_{dg}\Phi_1)(\Phi_1 e\Phi_1, \Phi_1 e\Phi_1))$ and $\mathscr{D}(\text{Mod-}H^0((\mathcal{C}_{dg}\Phi_1)(\Phi_1 e\Phi_1, \Phi_1 e\Phi_1)))$. And the latter one is triangle equivalent to $\mathscr{D}(\text{Mod-}e\Phi_1 e)$. Note that $e\Phi_1 e = A/I_{1,2}$ and $\mathrm{gl.dim}(A/I_{1,2}) < \infty$. Then, Φ_1^{op} and $(\Phi_1/\Phi_1 e\Phi_1)^{\mathrm{op}}$ are singularly equivalent, where

Φ_1^{op} and $(\Phi_1/\Phi_1 e\Phi_1)^{\mathrm{op}}$ are the opposite algebras of Φ_1 and $\Phi_1/\Phi_1 e\Phi_1$, respectively. Then, Φ_1 has finite global dimension if and only if so does $\Phi_1/\Phi_1 e\Phi_1$.

It is clear that

$$\Phi_1/\Phi_1 e\Phi_1 = \begin{pmatrix} A/I_{2,3} & I_{3,4}/I_{2,4} & \cdots & I_{3,n}/I_{2,n} \\ A/I_{2,3} & A/I_{2,4} & \cdots & I_{4,n}/I_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ A/I_{2,3} & A/I_{2,4} & \cdots & A/I_{2,n} \end{pmatrix}.$$

We write Φ_2 for $\Phi_1/\Phi_1 e\Phi_1$. Recursively, Φ_1 has finite global dimension if and only if so does $A/I_{n-1,n}$. Thus, we have $\mathrm{gl.dim}(\Phi_1) < \infty$. Hence, by Corollary 3.9, Δ satisfies ARC if and only if it holds for A since derived equivalences preserve ARC. $\square$

Example 5.3. (One-point extensions) Let A be an algebra, and let M be a left A-module. The one-point extension algebra $A[M]$ is defined to be the triangular matrix algebra

$$\begin{bmatrix} k & 0 \\ M & A \end{bmatrix}$$

If M has finite projective dimension, then ARC holds for $A[M]$ if and only if it holds for A. This follows immediately from Corollary 3.9.

Example 5.4. (Quotient algebras) Let A be an algebra, and let e be a primitive idempotent in A such that the multiplication map $Ae \otimes_k eA \to AeA$ is an isomorphism and $eAe \cong k$. The ideal AeA is called a heredity ideal in the literature. If furthermore the injective dimension of the right A-module eA is finite, then ARC holds for A if and only if it holds for A/AeA.

Indeed, it follows from [12] that $\mathscr{D}(A\text{-}\mathsf{Mod})$ admits a recollement

$$\begin{tikzcd} \mathscr{D}(A/AeA\text{-}\mathsf{Mod}) \arrow[r, "i_*"] & \mathscr{D}(A\text{-}\mathsf{Mod}) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \mathscr{D}(eAe\text{-}\mathsf{Mod}) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd} \tag{5.1}$$

where $i^* = A/AeA \otimes_A^{\mathbf{L}} -$, $i_* = A/AeA \otimes_{A/AeA}^{\mathbf{L}} -$, $i^! = \mathbf{RHom}_A(A/AeA, -)$, $j_! = Ae \otimes_{eAe}^{\mathbf{L}} -$, $j^* = eA \otimes_A^{\mathbf{L}} -$ and $j_* = \mathbf{RHom}_{eAe}(eA, -)$. Clearly, $\mathrm{gl.dim}\, eAe < \infty$ implies that $j^*A = eA \in \mathscr{K}^{\mathrm{b}}(eAe\text{-}\mathsf{proj})$, and $\mathrm{inj.dim}\, eA_A < \infty$ yields that $j_*(eAe) = \mathbf{RHom}_{eAe}(eA, eAe) = \mathbf{RHom}_k(eA, k) \in \mathscr{K}^{\mathrm{b}}(A\text{-}\mathsf{proj})$. Hence, by [1, Lemma 2.5 and Proposition 3.2], the recollement (5.1) can be extended two steps downwards. Similarly, by [1, Lemma 2.8 and Proposition 3.2], $\mathrm{proj.dim}\, Ae_{eAe} < \infty$ implies that (5.1) can be extended one step upwards. Therefore, (5.1) is completed to a ladder of height 4, and by Theorem 3.8, ARC holds for A if and only if it holds for A/AeA.

Example 5.5. (Derived discrete algebras) From [35], an algebra A is said to be *derived discrete* provided for every positive element $\mathbf{d} \in K_0(A)^{(\mathbb{Z})}$ there are only finitely many isomorphism classes of indecomposable objects X in $\mathscr{D}^{\mathrm{b}}(A\text{-}\mathsf{mod})$ of cohomology dimension vector $(\underline{\dim} H^p(X))_{p\in\mathbb{Z}} = \mathbf{d}$. Note that derived discrete algebras are of finite representation type, which were proved to satisfy ARC. Here, we apply our main theorem to reprove that ARC holds for all derived discrete algebras.

From [6,35], a basic connected derived discrete algebra A is derived equivalent to either a piecewise hereditary algebra of Dynkin type, or a bound quiver algebra $\Lambda(r,n,m)$ given by

$$\begin{tikzcd}
 & & & & 1 \arrow[r, "\alpha_1"] & \cdots \arrow[r, "\alpha_{n-r-2}"] & n-r-1 \arrow[dr, "\alpha_{n-r-1}"] & \\
(-m) \arrow[r, "\alpha_{-m}"] & \cdots \arrow[r, "\alpha_{-2}"] & (-1) \arrow[r, "\alpha_{-1}"] & 0 \arrow[ur, "\alpha_0"] & & & & n-r \arrow[dl, "\alpha_{n-r}"] \\
 & & & & n-1 \arrow[ul, "\alpha_{n-1}"] & \cdots \arrow[l, "\alpha_{n-2}"] & n-r+1 \arrow[l, "\alpha_{n-r+1}"] &
\end{tikzcd}$$

with the relations $\alpha_{n-1}\alpha_0, \alpha_{n-2}\alpha_{n-1}, \dots, \alpha_{n-r}\alpha_{n-r+1}$, where $1 \le r \le n$ and $m \ge 0$. Clearly, if $\mathrm{gl.dim}\, A < \infty$ then ARC always hold true, and if $\mathrm{gl.dim}\, A = \infty$ then A is derived equivalent to $\Lambda(n,n,m)$, which admits a series of infinite ladders, see [32, Lemma 17]. Note that the right terms of these ladders are k, and the left can be reduced to $\Lambda(n,n,0)$ consecutively. Hence, applying Theorem 3.8 shows that ARC holds for A if and only if it holds for $\Lambda(n,n,0)$, and the latter is known as 2-truncated cycle algebra, which is representation finite and then satisfies ARC by [3].

Data availability

Data will be made available on request.

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