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Gorenstein triangular matrix rings and category algebras

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ABSTRACT

We give conditions on when a triangular matrix ring is Gorenstein of a given selfinjective dimension. We apply the result to the category algebra of a finite EI category. In particular, we prove that for a finite EI category, its category algebra is 1-Gorenstein if and only if the given category is free and projective.

1. Introduction

Let R be a ring with a unit. Recall that R is *Gorenstein* if R is two-sided noetherian satisfying $\mathrm{id}_R R < \infty$ and $\mathrm{id} R_R < \infty$. Here, we use id to denote the injective dimension of a module. It is well known that for a Gorenstein ring R we have $\mathrm{id}_R R = \mathrm{id} R_R$; see [15, Lemma A]. Let $m \geq 0$. A Gorenstein ring R is *m-Gorenstein* if $\mathrm{id}_R R = \mathrm{id} R_R \leq m$. We observe that a 0-Gorenstein ring coincides with a quasi-Frobenius ring.

Let $n \geq 2$. Let $\Gamma = \begin{pmatrix} R_1 & M_{12} & \cdots & M_{1n} \\ & R_2 & \cdots & M_{2n} \\ & & \ddots & \vdots \\ & & & R_n \end{pmatrix}$ be an upper triangular matrix ring of order n, where each R_i is a ring and each M_{ij} is an R_i–R_j-bimodule together with bimodule morphisms $\psi_{ilj} : M_{il} \otimes_{R_l} M_{lj} \to M_{ij}$ satisfying

$$\psi_{ijt}(\psi_{ilj}(m_{il} \otimes m_{lj}) \otimes m_{jt}) = \psi_{ilt}(m_{il} \otimes \psi_{ljt}(m_{lj} \otimes m_{jt}))$$

for $1 \leq i < l < j < t \leq n$.

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For $1 \leq t \leq n-1$, we denote by $\Gamma_t = \begin{pmatrix} R_1 & M_{12} & \cdots & M_{1t} \\ & R_2 & \cdots & M_{2t} \\ & & \ddots & \vdots \\ & & & R_t \end{pmatrix}$ the ring given by the $t \times t$ leading principal submatrix of Γ. Denote the natural left Γ_t-module $\begin{pmatrix} M_{1,t+1} \\ \vdots \\ M_{t,t+1} \end{pmatrix}$ by M_t^*.

We will prove the following results, where the first statement is obtained by applying [3, Theorem 3.3] and [14, Lemma 2.6] repeatedly. We mention that the Gorensteinness of an upper triangular matrix ring is studied in [14] and [5].

Proposition 1.1. *Assume that $R_1, R_2, \cdots, R_n$ are quasi-Frobenius rings. Then*

(1) *The upper triangular matrix ring Γ is Gorenstein if and only if all the bimodules M_{ij} are finitely generated projective on both sides.*
(2) *The upper triangular matrix ring Γ is 1-Gorenstein if and only if all the bimodules M_{ij} are finitely generated projective on both sides, and each left Γ_t-module M_t^* is projective for $1 \leq t \leq n-1$.*

For the proof, (1) is a special case of Proposition 3.4. Thanks to (1), the "only if" part of (2) is a special case of Proposition 3.8(1), and the "if" part is a special case of Proposition 3.8(2). Here, we use the fact that a module over a quasi-Frobenius ring is projective provided that it has finite projective dimension.

Let k be a field and let $\mathscr{C}$ be a finite EI category. Here, the EI condition means that any endomorphism in $\mathscr{C}$ is an isomorphism. For an object x, we denote by $\mathrm{Aut}_{\mathscr{C}}(x)$ and $k\mathrm{Aut}_{\mathscr{C}}(x)$ the group of endomorphisms of x and the group algebra, respectively. We observe that for any two objects x and y in $\mathscr{C}$, $k\mathrm{Hom}_{\mathscr{C}}(x, y)$ is a $k\mathrm{Aut}_{\mathscr{C}}(y)$–$k\mathrm{Aut}_{\mathscr{C}}(x)$-bimodule. We say that a finite EI category $\mathscr{C}$ is *projective over* k if each bimodule $k\mathrm{Hom}_{\mathscr{C}}(x, y)$ is projective on both sides.

We denote by $k\mathscr{C}$ the category algebra of $\mathscr{C}$. We mention that category algebras play an important role in the representation theory of finite groups; see [10,11]. The following result is an application of Proposition 1.1(1), where we use the fact that the category algebra is isomorphic to a certain upper triangular matrix algebra.

Proposition 1.2. *Let k be a field and $\mathscr{C}$ be a finite EI category. Then the category algebra $k\mathscr{C}$ is Gorenstein if and only if $\mathscr{C}$ is projective over k.*

The concept of a finite *free* EI category is introduced in [8]. It is due to [8, Theorem 5.3] that the category algebra $k\mathscr{C}$ is hereditary if and only if $\mathscr{C}$ is a finite free EI category satisfying that the endomorphism groups of all objects have orders invertible in k. The following result is a Gorenstein analogue to [8, Theorem 5.3].

Theorem 1.3. *Let k be a field and $\mathscr{C}$ be a finite EI category. Then the category algebra $k\mathscr{C}$ is 1-Gorenstein if and only if $\mathscr{C}$ is a free EI category and projective over k.*

Indeed, we may deduce [8, Theorem 5.3] from Theorem 1.3, using the well-known fact that a finite dimensional algebra is hereditary if and only if it is 1-Gorenstein with finite global dimension; see Example 5.5.

This paper is organized as follows. In Section 2, we give an explicit description of projective modules and injective modules over an upper triangular matrix ring. In Section 3, we give conditions on when a triangular matrix ring is Gorenstein of a given selfinjective dimension, and prove Proposition 1.1. In Section 4, we give a new characterization of finite free EI categories in terms of the corresponding triangular matrix algebras; see Proposition 4.5. In Section 5, we prove Proposition 1.2 and Theorem 1.3.

2. Modules over triangular matrix rings

In this section, we describe explicitly projective modules and injective modules over an upper triangular matrix ring.

Let R_1 and R_2 be two rings, M_{12} an R_1–R_2-bimodule. We consider the corresponding upper triangular matrix ring $\Gamma = \begin{pmatrix} R_1 & M_{12} \\ 0 & R_2 \end{pmatrix}$.

Recall the description of left Γ-modules via column vectors. Let X_i be a left R_i-module, $i = 1, 2$, and let $\varphi_{12} : M_{12} \otimes_{R_2} X_2 \to X_1$ be a morphism of left R_1-modules. We define the left Γ-module structure on $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ by the following identity

$$\begin{pmatrix} r_1 & m_{12} \\ 0 & r_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} r_1x_1 + \varphi_{12}(m_{12} \otimes x_2) \\ r_2x_2 \end{pmatrix}.$$

We mention that the left Γ-module structure on $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ depends on the morphism φ_{12}. Indeed, every left Γ-module arises in this way; compare [1, III.2, Proposition 2.1]. A morphism $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \to \begin{pmatrix} X_1' \\ X_2' \end{pmatrix}$ of Γ-modules is denoted by $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$, where $f_i : X_i \to X_i'$ is an R_i-morphism satisfying

$$f_1 \circ \varphi_{12} = \varphi_{12}' \circ (\mathrm{Id}_{M_{12}} \otimes f_2). \tag{2.1}$$

Dually, we have the description of right Γ-modules via row vectors.

Let M be a left module over a ring R. We denote by $\mathrm{pd}_R M$ and $\mathrm{id}_R M$ the projective dimension and the injective dimension of M, respectively.

The following lemma is well-known; compare [1, III, Propositions 2.3 and 2.5] and [14, Lemma 1.2].

Lemma 2.1. *Let $\Gamma = \begin{pmatrix} R_1 & M_{12} \\ 0 & R_2 \end{pmatrix}$ be an upper triangular matrix ring, and let $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ be a left Γ-module as above. Then the following statements hold.*

(1) $\mathrm{pd}_{R_1} X_1 = \mathrm{pd}_\Gamma \begin{pmatrix} X_1 \\ 0 \end{pmatrix}$ *and* $\mathrm{id}_{R_2} X_2 = \mathrm{id}_\Gamma \begin{pmatrix} 0 \\ X_2 \end{pmatrix}$.

(2) $\mathrm{pd}_{R_2} X_2 \leq \mathrm{pd}_\Gamma \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ *and* $\mathrm{id}_{R_1} X_1 \leq \mathrm{id}_\Gamma \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$. □

Let $\Gamma = \begin{pmatrix} R_1 & M_{12} \\ 0 & R_2 \end{pmatrix}$ be as above. For each left R_1-module X_1, we associate two left Γ-modules $i_1(X_1) = \begin{pmatrix} X_1 \\ 0 \end{pmatrix}$ and $j_1(X_1) = \begin{pmatrix} X_1 \\ \mathrm{Hom}_{R_1}(M_{12}, X_1) \end{pmatrix}$, where the Γ-module structure on $j_1(X_1)$ is determined by the evaluation map $M_{12} \otimes_{R_2} \mathrm{Hom}_{R_1}(M_{12}, X_1) \to X_1$. For each left R_2-module X_2, we associate two left Γ-modules $i_2(X_2) = \begin{pmatrix} M_{12} \otimes_{R_2} X_2 \\ X_2 \end{pmatrix}$ and $j_2(X_2) = \begin{pmatrix} 0 \\ X_2 \end{pmatrix}$, where the Γ-module structure on $i_2(X_2)$ is determined by the identity map on $M_{12} \otimes_{R_2} X_2$.

The following result seems to be well known; compare [1, III, Proposition 2.5]. For completeness, we include a proof.

Lemma 2.2. *Let* $\Gamma = \begin{pmatrix} R_1 & M_{12} \\ 0 & R_2 \end{pmatrix}$ *be an upper triangular matrix ring. Then we have the following statements.*

(1) *A left* Γ*-module is projective if and only if it is isomorphic to* $i_1(P_1) \oplus i_2(P_2)$ *for some projective left* R_1*-module* P_1 *and projective left* R_2*-module* P_2*.*
(2) *A left* Γ*-module is injective if and only if it is isomorphic to* $j_1(I_1) \oplus j_2(I_2)$ *for some injective left* R_1*-module* I_1 *and injective left* R_2*-module* I_2*.*

Proof. We only prove (1). For the "if" part, we consider the subring $\Gamma' = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix}$ of Γ. Then $P' = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$ is a projective left Γ'-module. We observe that $i_1(P_1) \oplus i_2(P_2)$ isomorphic to $\Gamma \otimes_{\Gamma'} P'$. It follows that the left Γ-module $i_1(P_1) \oplus i_2(P_2)$ is projective.

For the "only if" part, let $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ be a projective left Γ-module. Then by Lemma 2.1(2) X_2 is a projective left R_2-module, and by the above $i_2(X_2)$ is a projective left Γ-module. Consider the nature projections $i_2(X_2) \overset{\pi_1}{\twoheadrightarrow} \begin{pmatrix} 0 \\ X_2 \end{pmatrix}$ and $X \overset{\pi_2}{\twoheadrightarrow} \begin{pmatrix} 0 \\ X_2 \end{pmatrix}$. Since $i_2(X_2)$ and X are projective left Γ-modules, we have two morphisms $i_2(X_2) \overset{\alpha}{\to} X$ and $X \overset{\beta}{\to} i_2(X_2)$ satisfying $\pi_1 = \pi_2 \circ \alpha$ and $\pi_2 = \pi_1 \circ \beta$. Therefore, $\pi_1 \circ \beta \circ \alpha = \pi_1$. By (2.1), we observe that $\beta \circ \alpha = \mathrm{Id}_{i_2(X_2)}$. It follows that α is a split monomorphism. In particular, X is isomorphic to $i_2(X_2) \oplus \mathrm{Coker}\alpha$. We observe that $\mathrm{Coker}\alpha$ is of the form $\begin{pmatrix} X_1' \\ 0 \end{pmatrix} = i_1(X_1')$ for some left R_1-module X_1'. Since $\mathrm{Coker}\alpha$ is a projective left Γ-module, we have that X_1' is a projective left R_1-module by Lemma 2.1(1). Then we are done. $\square$

We now extend the above results to an upper triangular matrix ring of an arbitrary order.

Let $n \geq 2$. Let R_i be a ring for $1 \leq i \leq n$, and let M_{ij} be an R_i–R_j-bimodule for $1 \leq i < j \leq n$. Let $\psi_{ilj} : M_{il} \otimes_{R_l} M_{lj} \to M_{ij}$ be morphisms of R_i–R_j-bimodules satisfying

$$\psi_{ijt}(\psi_{ilj}(m_{il} \otimes m_{lj}) \otimes m_{jt}) = \psi_{ilt}(m_{il} \otimes \psi_{ljt}(m_{lj} \otimes m_{jt}))$$

for $1 \leq i < l < j < t \leq n$.

Then we have the corresponding $n \times n$ upper triangular matrix ring $\Gamma = \begin{pmatrix} R_1 & M_{12} & \cdots & M_{1n} \\ & R_2 & \cdots & M_{2n} \\ & & \ddots & \vdots \\ & & & R_n \end{pmatrix}$. The elements of Γ, denoted by (m_{ij}), are $n \times n$ upper triangular matrices, where $m_{ij} \in M_{ij}$ for $i < j$ and $m_{ii} \in R_i$. The multiplication is induced by those morphisms ψ_{ilj}. We write $\psi_{ilj}(m_{il} \otimes m_{lj})$ as $m_{il}m_{lj}$.

We describe left Γ-modules via column vectors. Let X_i be a left R_i-module for $1 \leq i \leq n$, and $\varphi_{jl} : M_{jl} \otimes_{R_l} X_l \to X_j$ be a morphism of left R_j-modules satisfying

$$\varphi_{ij} \circ (\mathrm{Id}_{M_{ij}} \otimes \varphi_{jl}) = \varphi_{il} \circ (\psi_{ijl} \otimes \mathrm{Id}_{X_l})$$

for $1 \leq i < j < l \leq n$.

Set $X = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}$. Denote the elements of X by (x_k), where $x_k \in X_k$ for $1 \leq k \leq n$. We write $\varphi_{ij}(m_{ij} \otimes x_j)$ as $m_{ij}x_j$. Then we define the left Γ-module structure on X by the following identity

$$(m_{ij})(x_k) = (\sum_{l=i}^{n} m_{il}x_l).$$

We recall that m_{ij}'s are defined only for $i \leq j$, therefore, the range of the summation on the right side is from i to n. Indeed, every left Γ-module arises in this way.

Dually, we have the description of right Γ-modules via row vectors.

Let Γ be an upper triangular matrix ring of order n as above. We consider the subring $\Gamma^D = \mathrm{diag}(R_1, \cdots, R_n)$ of Γ consisting of diagonal matrices. Observe that Γ^D is isomorphic to the direct product ring $\prod_{i=1}^n R_i$. We view an R_i-module as a Γ^D-module via the projection $\Gamma^D \twoheadrightarrow R_i$. We denote the i-th column of Γ by C_i which is a Γ–R_i-bimodule, and denote the i-th row of Γ by H_i which is an R_i–Γ-bimodule.

Let $1 \leq t \leq n$, and let A be a left R_t-module. We consider the Γ-module $i_t(A) = C_t \otimes_{R_t} A$. Observe an isomorphism

$$\Gamma \otimes_{\Gamma^D} A \xrightarrow{\sim} i_t(A) \tag{2.2}$$

of Γ-modules sending $(m_{ij}) \otimes a$ to $(m_{it}) \otimes a$. We describe $i_t(A)$ as $\begin{pmatrix} M_{1t} \otimes_{R_t} A \\ \vdots \\ R_t \otimes_{R_t} A \\ \vdots \\ 0 \end{pmatrix}$, where the corresponding morphisms are $\varphi_{jl} = \psi_{jlt} \otimes \mathrm{Id}_A$ for $l < t$, $\varphi_{jt} : M_{jt} \otimes_{R_t} (R_t \otimes_{R_t} A) \to M_{jt} \otimes_{R_t} A$ is the canonical isomorphism, and $\varphi_{jl} = 0$ for $l > t$.

We consider the Γ-module $j_t(A) = \mathrm{Hom}_{R_t}(H_t, A)$. Observe an isomorphism

$$\mathrm{Hom}_{\Gamma^D}(\Gamma, A) \xrightarrow{\sim} j_t(A) \tag{2.3}$$

of Γ-modules sending f to its restriction on H_t. We describe $j_t(A)$ as $\begin{pmatrix} 0 \\ \vdots \\ \mathrm{Hom}_{R_t}(R_t, A) \\ \vdots \\ \mathrm{Hom}_{R_t}(M_{tn}, A) \end{pmatrix}$, where the corresponding morphisms $\varphi_{jl} : M_{jl} \otimes_{R_l} \mathrm{Hom}_{R_t}(M_{tl}, A) \to \mathrm{Hom}_{R_t}(M_{tj}, A)$ are given by $\varphi_{jl}(m_{jl} \otimes f)(m_{tj}) = f(m_{tj}m_{jl})$ for $j > t$, $\varphi_{tl} : M_{tl} \otimes_{R_l} \mathrm{Hom}_{R_t}(M_{tl}, A) \to \mathrm{Hom}_{R_t}(R_t, A)$ is the evaluation map, and $\varphi_{jl} = 0$ for $j < t$.

The following results give an explicit description of projective modules and injective modules over an upper triangular matrix ring.

Proposition 2.3. *Let Γ be an upper triangular matrix ring of order n. Then we have the following statements.*

(1) *A left Γ-module is projective if and only if it is isomorphic to $\bigoplus_{t=1}^{n} i_t(P_t)$ for some projective left R_t-module P_t, $1 \leq t \leq n$.*

(2) *A left* Γ*-module is injective if and only if it is isomorphic to* $\bigoplus_{t=1}^{n} j_t(I_t)$ *for some injective left* R_t*-module* I_t, $1 \leq t \leq n$.

Proof. We only prove (1). The "if" part is obvious since i_t preserves projective modules by the isomorphism (2.2).

For the "only if" part, we use induction on n. If $n = 2$, it is Lemma 2.2(1). Assume that $n > 2$. Write $\Gamma = \begin{pmatrix} \Gamma_{n-1} & M^*_{n-1} \\ 0 & R_n \end{pmatrix}$, where Γ_{n-1} is the $(n-1) \times (n-1)$ leading principal submatrix of Γ and $M^*_{n-1} = \begin{pmatrix} M_{1n} \\ \vdots \\ M_{n-1n} \end{pmatrix}$ is a Γ_{n-1}–R_n-bimodule. Assume that X is a projective left Γ-module. Write $X = \begin{pmatrix} X' \\ X_n \end{pmatrix}$, where $X' = \begin{pmatrix} X_1 \\ \vdots \\ X_{n-1} \end{pmatrix}$ is a left Γ_{n-1}-module. By Lemma 2.2(1), $X \simeq i'_1(X'_1) \oplus i_n(P_n)$, where X'_1 is a projective left Γ_{n-1}-module and P_n is a projective left R_n-module. By induction, we have an isomorphism $X'_1 \simeq \bigoplus_{t=1}^{n-1} i_t(P_t)$ of Γ_{n-1}-modules, where P_t is a projective left R_t-module, $1 \leq t \leq n-1$. We identify $i'_1 i_t(P_t)$ with $i_t(P_t)$. This completes the proof. □

3. Gorenstein triangular matrix rings

In this section, we study Gorenstein upper triangular matrix rings. We give conditions such that the upper triangular matrix rings are Gorenstein with a prescribed selfinjective dimension.

Let R be a ring with a unit. Recall that R is *Gorenstein* if R is two-sided noetherian satisfying $\mathrm{id}_R R < \infty$ and $\mathrm{id} R_R < \infty$. It is well known that for a Gorenstein ring R, $\mathrm{id}_R R = \mathrm{id} R_R$; see [15, Lemma A]. Let $m \geq 0$. A Gorenstein ring R is *m-Gorenstein* if $\mathrm{id}_R R = \mathrm{id} R_R \leq m$. Recall that for any m-Gorenstein ring R and any left (or right) R-module X, $\mathrm{id}_R X < \infty$ if and only if $\mathrm{id}_R X \leq m$, if and only if $\mathrm{pd}_R X < \infty$, if and only if $\mathrm{pd}_R X \leq m$; see [7, Theorem 9].

The following result generalizes [14, Lemma 2.6] with a different proof.

Lemma 3.1. *Let* $\Gamma = \begin{pmatrix} R_1 & M_{12} \\ 0 & R_2 \end{pmatrix}$ *be a Gorenstein upper triangular matrix ring. Then the following are equivalent.*

(1) $\mathrm{id}(R_1)_{R_1} < \infty$.
(2) *The ring* R_1 *is Gorenstein.*
(3) $\mathrm{pd}_{R_1} M_{12} < \infty$.
(4) $\mathrm{id}_{R_2} R_2 < \infty$.
(5) *The ring* R_2 *is Gorenstein.*
(6) $\mathrm{pd}(M_{12})_{R_2} < \infty$.

Proof. We observe that R_1 and R_2 are two-sided noetherian rings, since they are isomorphic to certain quotient rings of Γ.

"(1) ⇒ (2)" We observe that $\mathrm{id}_\Gamma \begin{pmatrix} R_1 \\ 0 \end{pmatrix} < \infty$, since $\begin{pmatrix} R_1 \\ 0 \end{pmatrix}$ is a projective left Γ-module and Γ is Gorenstein. Lemma 2.1(2) implies that $\mathrm{id}_{R_1} R_1 < \infty$. Hence R_1 is Gorenstein.

"(2) ⇒ (3)" We observe that $\mathrm{id}_\Gamma \begin{pmatrix} M_{12} \\ R_2 \end{pmatrix} < \infty$, since $\begin{pmatrix} M_{12} \\ R_2 \end{pmatrix}$ is a projective left Γ-module and Γ is Gorenstein. Lemma 2.1(2) implies that $\mathrm{id}_{R_1} M_{12} < \infty$. Since R_1 is Gorenstein, we have $\mathrm{pd}_{R_1} M_{12} < \infty$.

"(3) ⇒ (4)" By Lemma 2.1(1), $\mathrm{pd}_\Gamma \begin{pmatrix} M_{12} \\ 0 \end{pmatrix} = \mathrm{pd}_{R_1} M_{12} < \infty$. Since Γ is Gorenstein, we have $\mathrm{id}_\Gamma \begin{pmatrix} M_{12} \\ 0 \end{pmatrix} < \infty$. Recall from the above that $\mathrm{id}_\Gamma \begin{pmatrix} M_{12} \\ R_2 \end{pmatrix} < \infty$. The following exact sequence of Γ-modules

$$0 \to \begin{pmatrix} M_{12} \\ 0 \end{pmatrix} \to \begin{pmatrix} M_{12} \\ R_2 \end{pmatrix} \to \begin{pmatrix} 0 \\ R_2 \end{pmatrix} \to 0$$

implies $\mathrm{id}_\Gamma \begin{pmatrix} 0 \\ R_2 \end{pmatrix} < \infty$. By Lemma 2.1(1) $\mathrm{id}_{R_2} R_2 < \infty$. Then we are done.

We observe that the opposite ring Γ^{op} of Γ is identified with the upper triangular matrix ring $\begin{pmatrix} R_2^{\mathrm{op}} & M_{12} \\ 0 & R_1^{\mathrm{op}} \end{pmatrix}$. Then "(4) ⇒ (5)" is similar to "(1) ⇒ (2)"; "(5) ⇒ (6)" is similar to "(2) ⇒ (3)"; "(6) ⇒ (1)" is similar to "(3) ⇒ (4)". □

Lemma 3.2. *(See [3, Theorem 3.3].) Let* $\Gamma = \begin{pmatrix} R_1 & M_{12} \\ 0 & R_2 \end{pmatrix}$ *be an upper triangular matrix ring. Assume that* R_1 *and* R_2 *are Gorenstein. Then* Γ *is Gorenstein if and only if* M_{12} *is finitely generated and has finite projective dimension on both sides.*

Let $n \geq 2$, and let $\Gamma = \begin{pmatrix} R_1 & M_{12} & \cdots & M_{1n} \\ & R_2 & \cdots & M_{2n} \\ & & \ddots & \vdots \\ & & & R_n \end{pmatrix}$ be an upper triangular matrix ring of order n.

Notation 3.3. Set $M_{1:} = \begin{pmatrix} M_{12}, \cdots, M_{1n} \end{pmatrix}$. We denote by $\Gamma_t = \begin{pmatrix} R_1 & M_{12} & \cdots & M_{1t} \\ & R_2 & \cdots & M_{2t} \\ & & \ddots & \vdots \\ & & & R_t \end{pmatrix}$ the ring given by the $t \times t$ leading principal submatrix of Γ for $1 \leq t \leq n-1$. We denote by Γ'_{n-1} the ring given by the $(n-1) \times (n-1)$ principal submatrix of Γ which leaves out the first row and the first column. Denote the natural left Γ_t-module $\begin{pmatrix} M_{1,t+1} \\ \vdots \\ M_{t,t+1} \end{pmatrix}$ by M_t^*.

The following result extends Lemma 3.2.

Proposition 3.4. *Let* Γ *be an upper triangular matrix ring of order* n *as above. Assume that all* R_i *are Gorenstein. Then* Γ *is Gorenstein if and only if all bimodules* M_{ij} *are finitely generated and have finite projective dimension on both sides.*

Proof of the "only if" part. Assume that Γ is Gorenstein. We use induction on n. The case $n = 2$ is due to Lemma 3.2. Assume that $n > 2$. Write $\Gamma = \begin{pmatrix} \Gamma_{n-1} & M_{n-1}^* \\ 0 & R_n \end{pmatrix} = \begin{pmatrix} R_1 & M_{1:} \\ 0 & \Gamma'_{n-1} \end{pmatrix}$. Then Γ_{n-1} and Γ'_{n-1} are

Gorenstein by Lemma 3.1, and M^*_{n-1} and $M_{1:}$ are finitely generated and have finite projective dimension on both sides by Lemma 3.2. By induction, all M_{ij} possibly except for M_{1n} are finitely generated and have finite projective dimension on both sides. Since M_{1n} is a direct summand of M^*_{n-1} as a right R_n-module, it is finitely generated as a right R_n-module, and $\mathrm{pd}(M_{1n})_{R_n} < \infty$. Since M_{1n} is a direct summand of $M_{1:}$ as a left R_1-module, it is finitely generated as a left R_1-module, and $\mathrm{pd}_{R_1} M_{1n} < \infty$. □

We prove the "if" part of Proposition 3.4 together with the following lemma.

Lemma 3.5. *Let Γ be an upper triangular matrix ring of order n. Assume that all bimodules M_{ij} are finitely generated and have finite projective dimension on both sides.*

(1) *Let $X = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}$ be a left Γ-module. If each X_i is finitely generated as a left R_i-module satisfying $\mathrm{pd}_{R_i} X_i < \infty$, then X is finitely generated as a left Γ-module satisfying $\mathrm{pd}_\Gamma X < \infty$.*

(2) *Let $Y = \begin{pmatrix} Y_1, \cdots, Y_n \end{pmatrix}$ be a right Γ-module. If each Y_i is finitely generated as a right R_i-module satisfying $\mathrm{pd}(Y_i)_{R_i} < \infty$, then Y is finitely generated as a right Γ-module satisfying $\mathrm{pd}Y_\Gamma < \infty$.*

Proof of the "if" part of Proposition 3.4 and Lemma 3.5. We use induction on n.

If $n = 2$, Lemma 3.2 implies that Γ is Gorenstein. Let $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ be a left Γ-module. The following exact sequence of Γ-modules

$$0 \to \begin{pmatrix} X_1 \\ 0 \end{pmatrix} \to X \to \begin{pmatrix} 0 \\ X_2 \end{pmatrix} \to 0$$

implies that X is finitely generated, since $\begin{pmatrix} X_1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ X_2 \end{pmatrix}$ are finitely generated. We have $\mathrm{id}_{R_2} X_2 < \infty$, since R_2 is Gorenstein and $\mathrm{pd}_{R_2} X_2 < \infty$. By Lemma 2.1(1), $\mathrm{id}_\Gamma \begin{pmatrix} 0 \\ X_2 \end{pmatrix} = \mathrm{id}_{R_2} X_2 < \infty$, hence $\mathrm{pd}_\Gamma \begin{pmatrix} 0 \\ X_2 \end{pmatrix} < \infty$. By Lemma 2.1(1), $\mathrm{pd}_\Gamma \begin{pmatrix} X_1 \\ 0 \end{pmatrix} = \mathrm{pd}_{R_1} X_1 < \infty$. Then by the above exact sequence, we have $\mathrm{pd}_\Gamma X < \infty$. The proof of Lemma 3.5(2) is similar.

Assume that $n > 2$. Write $\Gamma = \begin{pmatrix} \Gamma_{n-1} & M^*_{n-1} \\ 0 & R_n \end{pmatrix} = \begin{pmatrix} R_1 & M_{1:} \\ 0 & \Gamma'_{n-1} \end{pmatrix}$. Let X be a left Γ-module and Y be a right Γ-module. Write $X = \begin{pmatrix} X' \\ X_n \end{pmatrix}$ and $Y = (Y_1, Y')$, where $X' = \begin{pmatrix} X_1 \\ \vdots \\ X_{n-1} \end{pmatrix}$ and $Y' = \begin{pmatrix} Y_2, \cdots, Y_n \end{pmatrix}$. By induction, Γ_{n-1} and Γ'_{n-1} are Gorenstein, X' is finitely generated as a left Γ_{n-1}-module satisfying $\mathrm{pd}_{\Gamma_{n-1}} X' < \infty$, and Y' is finitely generated as a right Γ'_{n-1}-module satisfying $\mathrm{pd}(Y')_{\Gamma'_{n-1}} < \infty$. We consider the left Γ_{n-1}-module M^*_{n-1}. By induction in Lemma 3.5(1), M^*_{n-1} is finitely generated as a left Γ_{n-1}-module satisfying $\mathrm{pd}_{\Gamma_{n-1}} M^*_{n-1} < \infty$. We observe that M^*_{n-1} is finitely generated as a right R_n-module satisfying $\mathrm{pd}(M^*_{n-1})_{R_n} < \infty$. By Lemma 3.2, Γ is Gorenstein. This proves the "if" part of Proposition 3.4 in the general case.

For Lemma 3.5(1), write $X = \begin{pmatrix} X' \\ X_n \end{pmatrix}$, where by induction X' is finitely generated as a left Γ_{n-1}-module satisfying $\mathrm{pd}_{\Gamma_{n-1}} X' < \infty$, and X_n is finitely generated as a left R_n-module satisfying $\mathrm{pd}_{R_n} X_n < \infty$. Then X is finitely generated as a left Γ-module satisfying $\mathrm{pd}_\Gamma X < \infty$. The proof of Lemma 3.5(2) in this general case is similar. □

We give a characterization of left Γ-modules with finite projective dimension; compare [5, Proposition 2.8(1)].

Corollary 3.6. *Let Γ be a Gorenstein upper triangular matrix ring of order n with each R_i Gorenstein. Let $X = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}$ be a finitely generated left Γ-module. Then $\mathrm{pd}_\Gamma X < \infty$ if and only if $\mathrm{pd}_{R_i} X_i < \infty$ for each $1 \le i \le n$.*

Proof. The "if" part is due to Lemma 3.5(1). For the "only if" part, we only prove the case $n = 2$. The general case is proved by induction. Let $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ be a left Γ-module. By Lemma 2.1(2), $\mathrm{pd}_{R_2} X_2 \le \mathrm{pd}_\Gamma X < \infty$ and $\mathrm{id}_{R_1} X_1 \le \mathrm{id}_\Gamma X < \infty$. Then $\mathrm{pd}_{R_1} X_1 < \infty$ since R_1 is Gorenstein. □

The following results estimate the selfinjective dimension of an upper triangular matrix ring; compare [3, Remark 3.5].

Proposition 3.7. *Let $\Gamma = \begin{pmatrix} R_1 & M_{12} \\ 0 & R_2 \end{pmatrix}$ be an upper triangular matrix ring with R_1 and R_2 Gorenstein. Let $m, d_1, d_2 \ge 0$.*

(1) *If Γ is m-Gorenstein, then both R_1 and R_2 are m-Gorenstein. Moreover, $\mathrm{pd}_{R_1} M_{12} \le m - 1$ if $m \ge 1$.*
(2) *Assume that R_i is d_i-Gorenstein for $i = 1, 2$. If M_{12} is finitely generated and projective on both sides, then Γ is d-Gorenstein, where $d = \max\{d_1, d_2\}$ if $d_1 \ne d_2$, and $d = d_1 + 1$ if $d_1 = d_2$.*

Proof. (1) We have $\mathrm{id}_\Gamma \begin{pmatrix} R_1 \\ 0 \end{pmatrix} \le m$, since Γ is m-Gorenstein and $\begin{pmatrix} R_1 \\ 0 \end{pmatrix}$ is a projective Γ-module. By Lemma 2.1(2), $\mathrm{id}_{R_1} R_1 \le \mathrm{id}_\Gamma \begin{pmatrix} R_1 \\ 0 \end{pmatrix} \le m$. Then R_1 is m-Gorenstein. Recall that, by Lemma 3.1, $\mathrm{pd}_{R_1} M_{12} < \infty$. By Lemma 2.1(1), $\mathrm{pd}_\Gamma \begin{pmatrix} M_{12} \\ 0 \end{pmatrix} = \mathrm{pd}_{R_1} M_{12} < \infty$. Then the following exact sequence of Γ-modules

$$0 \to \begin{pmatrix} M_{12} \\ 0 \end{pmatrix} \to \begin{pmatrix} M_{12} \\ R_2 \end{pmatrix} \to \begin{pmatrix} 0 \\ R_2 \end{pmatrix} \to 0$$

implies $\mathrm{pd}_\Gamma \begin{pmatrix} 0 \\ R_2 \end{pmatrix} < \infty$, since $\begin{pmatrix} M_{12} \\ R_2 \end{pmatrix}$ is a projective Γ-module. Then we have $\mathrm{id}_\Gamma \begin{pmatrix} 0 \\ R_2 \end{pmatrix} \le m$ since Γ is m-Gorenstein. By Lemma 2.1(1), $\mathrm{id}_{R_2} R_2 = \mathrm{id}_\Gamma \begin{pmatrix} 0 \\ R_2 \end{pmatrix} \le m$. Then R_2 is m-Gorenstein. Moreover, by the above exact sequence, $\mathrm{pd}_\Gamma\begin{pmatrix} M_{12} \\ 0 \end{pmatrix} = 0$ or $\mathrm{pd}_\Gamma\begin{pmatrix} M_{12} \\ 0 \end{pmatrix} = \mathrm{pd}_\Gamma\begin{pmatrix} 0 \\ R_2 \end{pmatrix} - 1 \le m-1$ for $m \ge 1$. Hence by Lemma 2.1(1), $\mathrm{pd}_{R_1} M_{12} \le m-1$ for $m \ge 1$.

(2) By Lemma 3.2, Γ is Gorenstein. By the following exact sequence of left Γ-modules

$$0 \to \begin{pmatrix} R_1 \\ 0 \end{pmatrix} \to j_1(R_1) = \begin{pmatrix} R_1 \\ \mathrm{Hom}_{R_1}(M_{12}, R_1) \end{pmatrix} \to \begin{pmatrix} 0 \\ \mathrm{Hom}_{R_1}(M_{12}, R_1) \end{pmatrix} \to 0,$$

we have

$$\mathrm{id}_\Gamma\begin{pmatrix} R_1 \\ 0 \end{pmatrix} \le \max\left\{\mathrm{id}_\Gamma j_1(R_1), \mathrm{id}_\Gamma\begin{pmatrix} 0 \\ \mathrm{Hom}_{R_1}(M_{12}, R_1) \end{pmatrix} + 1\right\}.$$

Observe that for any left R_1-module X, $\mathrm{id}_{R_2}(\mathrm{Hom}_{R_1}(M_{12}, X)) \le \mathrm{id}_{R_1} X$ since ${}_{R_1}(M_{12})_{R_2}$ is projective on both sides. Then we have $\mathrm{id}_{R_2}(\mathrm{Hom}_{R_1}(M_{12}, R_1)) \le \mathrm{id}_{R_1} R_1 \le d_1$. We have $\mathrm{id}_{R_2}(\mathrm{Hom}_{R_1}(M_{12}, R_1)) \le d_2$, since R_2 is d_2-Gorenstein. By Lemma 2.1(1), $\mathrm{id}_\Gamma\begin{pmatrix} 0 \\ \mathrm{Hom}_{R_1}(M_{12}, R_1) \end{pmatrix} = \mathrm{id}_{R_2}(\mathrm{Hom}_{R_1}(M_{12}, R_1)) \le \min\{d_1, d_2\}$. The exact functor j_1 preserves injective modules. Then we have $\mathrm{id}_\Gamma j_1(R_1) \le \mathrm{id}_{R_1} R_1 \le d_1$. Hence $\mathrm{id}_\Gamma\begin{pmatrix} R_1 \\ 0 \end{pmatrix} \le \max\{d_1, \min\{d_1, d_2\} + 1\} \le d$. We have $\mathrm{id}_\Gamma\begin{pmatrix} M_{12} \\ 0 \end{pmatrix} \le d$, since M_{12} is a finitely generated projective left R_1-module.

By Lemma 2.1(1), $\mathrm{id}_\Gamma\begin{pmatrix} 0 \\ R_2 \end{pmatrix} = \mathrm{id}_{R_2} R_2 \le d_2 \le d$ since R_2 is d_2-Gorenstein. Then the following exact sequence of Γ-modules

$$0 \to \begin{pmatrix} M_{12} \\ 0 \end{pmatrix} \to \begin{pmatrix} M_{12} \\ R_2 \end{pmatrix} \to \begin{pmatrix} 0 \\ R_2 \end{pmatrix} \to 0$$

implies $\mathrm{id}_\Gamma\begin{pmatrix} M_{12} \\ R_2 \end{pmatrix} \le d$. Hence $\mathrm{id}_\Gamma \Gamma \le d$ since $\Gamma = \begin{pmatrix} R_1 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} M_{12} \\ R_2 \end{pmatrix}$ as left Γ-modules. Then Γ is d-Gorenstein. □

The following results estimate the selfinjective dimension of an upper triangular matrix ring in general case. Recall the Γ_t-module M_t^* from Notation 3.3.

Proposition 3.8. *Let Γ be an upper triangular matrix ring of order n with each R_i Gorenstein. Let $m \ge 0$ and $d_i \ge 0$ for each i.*

(1) *If Γ is m-Gorenstein, then each R_i is m-Gorenstein. Moreover, $\mathrm{pd}_{\Gamma_t} M_t^* \le m-1$ if $m \ge 1$ for $1 \le t \le n-1$.*
(2) *Assume that R_i is d_i-Gorenstein for $1 \le i \le n$. If all bimodules M_{ij} are finitely generated and projective on both sides, and each M_t^* is a projective left Γ_t-module for $1 \le t \le n-1$, then Γ is d-Gorenstein, where $d = \max\{d_1, d_2, \cdots, d_n\} + 1$.*

Proof. (1) We use induction on n. The case $n = 2$ is due to Proposition 3.7(1). Assume that $n > 2$. Write $\Gamma = \begin{pmatrix} \Gamma_{n-1} & M_{n-1}^* \\ 0 & R_n \end{pmatrix}$. By Lemma 3.1, Γ_{n-1} is Gorenstein. Then by Proposition 3.7(1), we infer that Γ_{n-1} and R_n are m-Gorenstein, and $\mathrm{pd}_{\Gamma_{n-1}} M_{n-1}^* \le m-1$ if $m \ge 1$. By induction, we are done.

(2) We use induction on n. The case $n = 2$ is due to Proposition 3.7(2). Assume that $n > 2$. Write $\Gamma = \begin{pmatrix} \Gamma_{n-1} & M^*_{n-1} \\ 0 & R_n \end{pmatrix}$. By induction, Γ_{n-1} is d'-Gorenstein, where $d' = \max\{d_1, d_2, \cdots, d_{n-1}\} + 1$. Since M^*_{n-1} is a projective left Γ_{n-1}-module and a projective right R_n-module, we have that Γ is d''-Gorenstein by Proposition 3.7(2), where $d'' = \max\{d', d_n\}$ if $d' \neq d_n$, and $d'' = d' + 1$ if $d' = d_n$. In particular, we observe that $d'' \leq \max\{d_1, d_2, \cdots, d_n\} + 1 = d$. $\square$

4. Free EI categories

In this section, we give a new characterization of finite free EI categories in terms of the corresponding triangular matrix algebras.

Let k be a field. Let $\mathscr{C}$ be a finite category, that is, it has only finitely many morphisms, and consequently it has only finitely many objects. Denote by $\mathrm{Mor}\mathscr{C}$ the finite set of all morphisms in $\mathscr{C}$. The *category algebra* $k\mathscr{C}$ of $\mathscr{C}$ is defined as follows: $k\mathscr{C} = \bigoplus_{\alpha \in \mathrm{Mor}\mathscr{C}} k\alpha$ as a k-vector space and the product $*$ is given by the rule

$$\alpha * \beta = \begin{cases} \alpha \circ \beta, & \text{if } \alpha \text{ and } \beta \text{ can be composed in } \mathscr{C}; \\ 0, & \text{otherwise.} \end{cases}$$

The unit is given by $1_{k\mathscr{C}} = \sum_{x \in \mathrm{Obj}\mathscr{C}} \mathrm{Id}_x$, where Id_x is the identity endomorphism of an object x in $\mathscr{C}$.

Let $\mathscr{C}$ be a finite category. We recall that a module over $k\mathscr{C}$ is identified with a functor from $\mathscr{C}$ to the category of finite dimensional k-vector spaces; see [11, Proposition 2.1]. If $\mathscr{C}$ and $\mathscr{D}$ are two equivalent finite categories, then $k\mathscr{C}$ and $k\mathscr{D}$ are Morita equivalent; see [11, Proposition 2.2]. In particular, $k\mathscr{C}$ is Morita equivalent to $k\mathscr{C}_0$, where $\mathscr{C}_0$ is any skeleton of $\mathscr{C}$. So we may assume that $\mathscr{C}$ is *skeletal*, that is, for any two distinct objects x and y in $\mathscr{C}$, x is not isomorphic to y.

The category $\mathscr{C}$ is called a *finite EI category* provided that all endomorphisms in $\mathscr{C}$ are isomorphisms. In particular, $\mathrm{Hom}_{\mathscr{C}}(x, x) = \mathrm{Aut}_{\mathscr{C}}(x)$ is a finite group for any object x in $\mathscr{C}$.

In what follows, we assume that $\mathscr{C}$ is a finite EI category which is skeletal.

Let $\mathscr{C}$ have n objects with $n \geq 2$. We assume that $\mathrm{Obj}\mathscr{C} = \{x_1, x_2, \cdots, x_n\}$ satisfying $\mathrm{Hom}_{\mathscr{C}}(x_i, x_j) = \emptyset$ if $i < j$. Let $M_{ij} = k\mathrm{Hom}_{\mathscr{C}}(x_j, x_i)$. Write $R_i = M_{ii}$. We observe that $R_i = (\mathrm{Id}_{x_i})k\mathscr{C}(\mathrm{Id}_{x_i}) = k\mathrm{Aut}_{\mathscr{C}}(x_i)$ is a group algebra.

Then M_{ij} is naturally an R_i–R_j-bimodule, and we have a morphism of R_i–R_j-bimodules $\psi_{ilj} : M_{il} \otimes_{R_l} M_{lj} \to M_{ij}$ which is induced by the composition of morphisms in $\mathscr{C}$.

Notation 4.1. The category algebra $k\mathscr{C}$ is isomorphic to the corresponding upper triangular matrix algebra $\Gamma_{\mathscr{C}} = \begin{pmatrix} R_1 & M_{12} & \cdots & M_{1n} \\ & R_2 & \cdots & M_{2n} \\ & & \ddots & \vdots \\ & & & R_n \end{pmatrix}$. Let Γ_t be the algebra given by the $t \times t$ leading principal submatrix of $\Gamma_{\mathscr{C}}$. Denote the left Γ_t-module $\begin{pmatrix} M_{1,t+1} \\ \vdots \\ M_{t,t+1} \end{pmatrix}$ by M^*_t, for $1 \leq t \leq n-1$.

Definition 4.2. Let $\mathscr{C}$ be a finite EI category with $\mathrm{Obj}\mathscr{C} = \{x_1, x_2, \cdots, x_n\}$ satisfying $\mathrm{Hom}_{\mathscr{C}}(x_i, x_j) = \emptyset$ if $i < j$. We say that $\mathscr{C}$ is *projective over* k if each $M_{ij} = k\mathrm{Hom}_{\mathscr{C}}(x_j, x_i)$ is a projective left R_i-module and a projective right R_j-module for $1 \leq i < j \leq n$.

Let G be a finite group. We assume that G has a left action on a finite set X. For any $x \in X$, denote its stabilizer by $\mathrm{Stab}(x) = \{g \in G \mid g.x = x\}$. The vector space kX is a natural kG-module.

The following result is well known, which can be deduced from [6, II.5 Theorem 6]. We give an elementary argument for completeness; compare the third paragraph of the proof of [10, Theorem 2.5].

Lemma 4.3. *The kG-module kX is projective if and only if the order of each stabilizer* $\mathrm{Stab}(x)$ *is invertible in k.*

Proof. We may assume that the action on X is transitive. Take $x \in X$, we have $X \simeq G/\mathrm{Stab}(x)$. Then we have an isomorphism $kX \simeq k(G/\mathrm{Stab}(x))$ of kG-modules. We observe the following Maschke-type result: for any subgroup H of G, the canonical projection $kG \twoheadrightarrow k(G/H)$ of kG-modules splits if and only if the order of H is invertible in k. Then the lemma follows immediately. $\square$

Let $\mathscr{C}$ be a finite EI category and $\alpha \in \mathrm{Hom}_{\mathscr{C}}(x, y)$. We call $L_\alpha = \{g \in \mathrm{Aut}_{\mathscr{C}}(y) \mid g \circ \alpha = \alpha\}$ the *left stabilizer* of α, and $R_\alpha = \{h \in \mathrm{Aut}_{\mathscr{C}}(x) \mid \alpha \circ h = \alpha\}$ the *right stabilizer* of α. Then we have the following immediate consequence of Lemma 4.3.

Corollary 4.4. *Let $\mathscr{C}$ be a finite EI category. Then $\mathscr{C}$ is projective if and only if for any $\alpha \in \mathrm{Mor}\mathscr{C}$, the orders of L_α and R_α are invertible in k.* $\square$

Let $\mathscr{C}$ be a finite EI category. Recall from [8, Definition 2.3] that a morphism $x \xrightarrow{\alpha} y$ in $\mathscr{C}$ is *unfactorizable* if α is not an isomorphism and whenever it has a factorization as a composite $x \xrightarrow{\beta} z \xrightarrow{\gamma} y$, then either β or γ is an isomorphism. Let $x \xrightarrow{\alpha} y$ in $\mathscr{C}$ be an unfactorizable morphism. Then $h \circ \alpha \circ g$ is also unfactorizable for every $h \in \mathrm{Aut}_{\mathscr{C}}(y)$ and every $g \in \mathrm{Aut}_{\mathscr{C}}(x)$; see [8, Proposition 2.5]. Let $x \xrightarrow{\alpha} y$ in $\mathscr{C}$ be a morphism with $x \neq y$. Then it has a decomposition $x = x_0 \xrightarrow{\alpha_1} x_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} x_n = y$ with all α_i unfactorizable; see [8, Proposition 2.6].

Following [8, Definition 2.7], we say that a finite EI category $\mathscr{C}$ satisfies the Unique Factorization Property (UFP), if whenever a non-isomorphism α has two decompositions into unfactorizable morphisms:

$$x = x_0 \xrightarrow{\alpha_1} x_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_m} x_m = y$$

and

$$x = y_0 \xrightarrow{\beta_1} y_1 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_n} y_n = y,$$

then $m = n$, $x_i = y_i$, and there are $h_i \in \mathrm{Aut}_{\mathscr{C}}(x_i)$, $1 \le i \le n-1$ such that $\beta_1 = h_1 \circ \alpha_1$, $\beta_2 = h_2 \circ \alpha_2 \circ h_1^{-1}$, $\cdots$, $\beta_{n-1} = h_{n-1} \circ \alpha_{n-1} \circ h_{n-2}^{-1}$, $\beta_n = \alpha_n \circ h_{n-1}^{-1}$.

Let $\mathscr{C}$ be a finite EI category. Following [9, Section 6], we say that $\mathscr{C}$ is a finite *free* EI category if it satisfies the UFP. This is an equivalent characterization of finite free EI categories in [8, Proposition 2.8].

Let $\Gamma = \Gamma_{\mathscr{C}}$ be the corresponding upper triangular matrix algebra of $\mathscr{C}$. We recall the Γ_t-module M_t^* from Notation 4.1 for each $1 \le t \le n-1$. We have the following characterization of finite free EI categories.

Proposition 4.5. *Let $\mathscr{C}$ be a finite skeletal EI category with* $\mathrm{Obj}\mathscr{C} = \{x_1, x_2, \cdots, x_n\}$ *satisfying* $\mathrm{Hom}_{\mathscr{C}}(x_i, x_j) = \emptyset$ *if $i < j$. Assume that $\mathscr{C}$ is projective. Then $\mathscr{C}$ is a free EI category if and only if each M_t^* is a projective left Γ_t-module for $1 \le t \le n-1$.*

Before giving the proof of the proposition, we make some preparations.

Definition 4.6. Let $\mathscr{C}$ be a finite EI category and $x \in \mathrm{Obj}\mathscr{C}$. We say that the EI category $\mathscr{C}$ is *free from* x if whenever an arbitrary non-isomorphism $x \stackrel{\alpha}{\to} y$ in $\mathscr{C}$ has two decompositions $x \stackrel{\alpha_1}{\to} z_1 \stackrel{\alpha_2}{\to} y$ and $x \stackrel{\beta_1}{\to} z_2 \stackrel{\beta_2}{\to} y$ with α_1 and β_1 unfactorizable, then $z_1 = z_2$ and there is an endomorphism $h \in \mathrm{Aut}_{\mathscr{C}}(z_1)$ such that $\beta_1 = h \circ \alpha_1$ and $\beta_2 = \alpha_2 \circ h^{-1}$.

Lemma 4.7. *Let $\mathscr{C}$ be a finite EI category. Then $\mathscr{C}$ is a free EI category if and only if $\mathscr{C}$ is free from any object.*

Proof. The "only if" part is trivial. For the "if" part, assume that the EI category $\mathscr{C}$ is free from any object. Let $x \stackrel{\alpha}{\to} y$ be a non-isomorphism in $\mathscr{C}$. Assume that α has two decompositions into unfactorizable morphisms:

$$x = x_0 \stackrel{\alpha_1}{\to} x_1 \stackrel{\alpha_2}{\to} \cdots \stackrel{\alpha_m}{\to} x_m = y$$

and

$$x = y_0 \stackrel{\beta_1}{\to} y_1 \stackrel{\beta_2}{\to} \cdots \stackrel{\beta_n}{\to} y_n = y.$$

Since $\mathscr{C}$ is free from x, we have $x_1 = y_1$, and there is an endomorphism $h_1 \in \mathrm{Aut}_{\mathscr{C}}(x_1)$ such that $\beta_1 = h_1 \circ \alpha_1$ and $\alpha_m \circ \cdots \alpha_2 = \beta_n \circ \cdots \beta_2 \circ h_1$. We continue this process. We obtain that $m = n$, $x_i = y_i$, and there are $h_i \in \mathrm{Aut}_{\mathscr{C}}(x_i)$, $1 \leq i \leq m-1$ such that $\beta_1 = h_1 \circ \alpha_1$, $\beta_2 = h_2 \circ \alpha_2 \circ h_1^{-1}$, $\cdots$, $\beta_{m-1} = h_{m-1} \circ \alpha_{m-1} \circ h_{m-2}^{-1}$, $\beta_m = \alpha_m \circ h_{m-1}^{-1}$. Then $\mathscr{C}$ is free. □

Let $W_{il} \subseteq \mathrm{Hom}_{\mathscr{C}}(x_l, x_i)$ and $T_{lj} \subseteq \mathrm{Hom}_{\mathscr{C}}(x_j, x_l)$ be subsets. Denote the subset $W_{il} \circ T_{lj} = \{f \circ g \mid f \in W_{il} \text{ and } g \in T_{lj}\} \subseteq \mathrm{Hom}_{\mathscr{C}}(x_j, x_i)$.

Notation 4.8. Set $\mathrm{Hom}^0_{\mathscr{C}}(x_j, x_i) = \{\alpha \in \mathrm{Hom}_{\mathscr{C}}(x_j, x_i) \mid \alpha \text{ is unfactorizable}\}$. Denote $M^0_{ij} = k\mathrm{Hom}^0_{\mathscr{C}}(x_j, x_i)$. Then M^0_{ij} is an R_i–R_j-subbimodule of M_{ij}. Moreover,

$$M_{ij} = M^0_{ij} \oplus (\sum_{l=i+1}^{j-1} k(\mathrm{Hom}_{\mathscr{C}}(x_l, x_i) \circ \mathrm{Hom}^0_{\mathscr{C}}(x_j, x_l)))$$

as an R_i–R_j-bimodule.

Lemma 4.9. *Let $\mathscr{C}$ be a finite EI category with $\mathrm{Obj}\mathscr{C} = \{x_1, x_2, \cdots, x_n\}$ satisfying $\mathrm{Hom}_{\mathscr{C}}(x_i, x_j) = \emptyset$ if $i < j$. Assume that $\mathscr{C}$ is free from x_j. Then for any $1 \leq i < j$, we have*

$$\mathrm{Hom}_{\mathscr{C}}(x_j, x_i) = \bigsqcup_{l=i}^{j-1} (\mathrm{Hom}_{\mathscr{C}}(x_l, x_i) \circ \mathrm{Hom}^0_{\mathscr{C}}(x_j, x_l)),$$

where the right hand side is a disjoint union.

Proof. Recall that the category $\mathscr{C}$ is free from x_j. Then we have

$$\mathrm{Hom}_{\mathscr{C}}(x_{l_1}, x_i) \circ \mathrm{Hom}^0_{\mathscr{C}}(x_j, x_{l_1}) \cap \mathrm{Hom}_{\mathscr{C}}(x_{l_2}, x_i) \circ \mathrm{Hom}^0_{\mathscr{C}}(x_j, x_{l_2}) = \phi$$

for $l_1 \neq l_2$. Since every morphism can be decomposed as a composition of unfactorizable morphisms, we have the required equation. □

We observe that there is a surjective morphism

$$\xi_{ilj} : M_{il} \otimes_{R_l} M^0_{lj} \longrightarrow k(\mathrm{Hom}_{\mathscr{C}}(x_l, x_i) \circ \mathrm{Hom}^0_{\mathscr{C}}(x_j, x_l)) \tag{4.1}$$

of R_i–R_j-bimodules sending $\beta \otimes \alpha$ to $\beta \circ \alpha$ for $i < l < j$.

We recall the Γ_{n-1}-module $i_t(A)$ for each $1 \le t \le n-1$ and each R_t-module A; see (2.2). Then we have the following natural surjective morphisms

$$\Phi_i : M^0_{in} \oplus (\bigoplus_{l=i+1}^{n-1} (M_{il} \otimes_{R_l} M^0_{ln})) \longrightarrow M_{in} \tag{4.2}$$

of R_i–R_n-bimodules induced by ξ_{iln} for $1 \le i < l \le n-1$, and

$$\Phi : \bigoplus_{t=1}^{n-1} i_t(M^0_{tn}) \longrightarrow M^*_{n-1} \tag{4.3}$$

of Γ_{n-1}–R_n-bimodules induced by Φ_i for $1 \le i \le n-1$.

Lemma 4.10. *Let $\mathscr{C}$ be a finite EI category with $\mathrm{Obj}\mathscr{C} = \{x_1, x_2, \cdots, x_n\}$ satisfying $\mathrm{Hom}_{\mathscr{C}}(x_i, x_j) = \emptyset$ if $i < j$. Assume that $\mathscr{C}$ is projective. Then the above surjective morphism Φ is a projective cover of the left Γ_{n-1}-module M^*_{n-1}.*

Proof. Since the category $\mathscr{C}$ is projective, we have that M_{tn} is a projective left R_t-module for each $1 \le t \le n-1$. Then each M^0_{tn} is a projective left R_t-module, since it is a direct summand of M_{tn}; see Notation 4.8. By Proposition 2.3(1), $\bigoplus_{t=1}^{n-1} i_t(M^0_{tn})$ is a projective left Γ_{n-1}-module.

To prove that Φ is a projective cover, it suffices to show that $\mathrm{top}(\bigoplus_{t=1}^{n-1} i_t(M^0_{tn}))$ and $\mathrm{top}(M^*_{n-1})$ are isomorphic. Here, we write $\mathrm{top}X = X/\mathrm{rad}X$ for a module X, where $\mathrm{rad}X$ denotes the radical of X. Recall $\mathrm{rad}(\Gamma_{n-1}) = \begin{pmatrix} \mathrm{rad}(R_1) & M_{12} & \cdots & M_{1,n-1} \\ & \mathrm{rad}(R_2) & \cdots & M_{2,n-1} \\ & & \ddots & \vdots \\ & & & \mathrm{rad}(R_{n-1}) \end{pmatrix}$. By $\mathrm{rad}(\bigoplus_{t=1}^{n-1} i_t(M^0_{tn})) = \mathrm{rad}(\Gamma_{n-1})(\bigoplus_{t=1}^{n-1} i_t(M^0_{tn}))$, we compute that the i-th component of $\mathrm{top}(\bigoplus_{t=1}^{n-1} i_t(M^0_{tn}))$ is isomorphic to $M^0_{in}/\mathrm{rad}(R_i)M^0_{in}$. By a similar calculation, we have that the i-th component of $\mathrm{top}(M^*_{n-1})$ is isomorphic to $M^0_{in}/\mathrm{rad}(R_i)M^0_{in}$. Then we have the required isomorphism. □

Lemma 4.11. *Let $\mathscr{C}$ be a finite EI category with $\mathrm{Obj}\mathscr{C} = \{x_1, x_2, \cdots, x_n\}$ satisfying $\mathrm{Hom}_{\mathscr{C}}(x_i, x_j) = \emptyset$ if $i < j$. Assume that $\mathscr{C}$ is projective. Then the following are equivalent.*

(1) *The category $\mathscr{C}$ is free from x_n.*
(2) *All the surjective morphisms Φ_i are isomorphisms.*
(3) *The surjective morphism Φ is an isomorphism.*
(4) *The left Γ_{n-1}-module M^*_{n-1} is projective.*

Proof. "(1) ⇒ (2)" Since the category $\mathscr{C}$ is free from x_n, by Notation 4.8 and Lemma 4.9 we have

$$M_{in} = M^0_{in} \oplus (\bigoplus_{l=i+1}^{n-1} k(\mathrm{Hom}_{\mathscr{C}}(x_l, x_i) \circ \mathrm{Hom}^0_{\mathscr{C}}(x_n, x_l))).$$

Since Φ_i is induced by ξ_{iln}, we only need to prove that ξ_{iln} is an isomorphism for each $i < l < n$. Indeed, since $\mathscr{C}$ is free from x_n, we have $\beta' \circ \alpha' = \beta \circ \alpha$ in $\mathrm{Hom}_{\mathscr{C}}(x_l, x_i) \circ \mathrm{Hom}^0_{\mathscr{C}}(x_n, x_l)$ if and only if $\beta' = \beta \circ g$ and $\alpha' = g^{-1} \circ \alpha$ for some $g \in \mathrm{Aut}_{\mathscr{C}}(x_l)$. Then we have a well-defined morphism

$$\eta_{iln} : k(\mathrm{Hom}_{\mathscr{C}}(x_l, x_i) \circ \mathrm{Hom}^0_{\mathscr{C}}(x_n, x_l)) \longrightarrow M_{il} \otimes_{R_l} M^0_{ln}$$

of R_i–R_n-bimodules sending $\beta \circ \alpha$ to $\beta \otimes \alpha$. It is directly verify that ξ_{iln} and η_{iln} are mutually inverse. Then we have the required isomorphisms.

"(2) $\Rightarrow$ (1)" Since each Φ_i is an isomorphism for $1 \le i \le n-1$, we have

$$\mathrm{Hom}_{\mathscr{C}}(x_{l_1}, x_i) \circ \mathrm{Hom}^0_{\mathscr{C}}(x_n, x_{l_1}) \cap \mathrm{Hom}_{\mathscr{C}}(x_{l_2}, x_i) \circ \mathrm{Hom}^0_{\mathscr{C}}(x_n, x_{l_2}) = \phi$$

for $i < l_1 \neq l_2 < n$, which implies that $\mathscr{C}$ is free from x_n.

"(2) $\Leftrightarrow$ (3)" It is obvious, since Φ is induced by $\Phi_1, \cdots, \Phi_{n-1}$.

"(3) $\Leftrightarrow$ (4)" Apply Lemma 4.10. □

Proof of Proposition 4.5. Assume that $\mathscr{C}$ is projective. For each $1 \le t \le n-1$, we consider the full subcategory $\mathscr{C}_t$ of $\mathscr{C}$ with $\mathrm{Obj}\mathscr{C}_t = \{x_1, \cdots, x_t\}$. We observe that $\mathscr{C}_t$ is free from x_t if and only if $\mathscr{C}$ is free from x_t. Then by Lemma 4.11, we have that $\mathscr{C}$ is free from x_t if and only if M^*_t is a projective left Γ_t-module. By Lemma 4.7, we are done. □

5. The main results

In this section, we give a necessary and sufficient condition on when the category algebra $k\mathscr{C}$ of a finite EI category $\mathscr{C}$ is Gorenstein, and when $k\mathscr{C}$ is 1-Gorenstein.

Throughout this section, when the category $\mathscr{C}$ is skeletal, we assume that $\mathrm{Obj}\mathscr{C} = \{x_1, x_2, \cdots, x_n\}$ satisfying $\mathrm{Hom}_{\mathscr{C}}(x_i, x_j) = \emptyset$ if $i < j$.

Proposition 5.1. *Let k be a field and $\mathscr{C}$ be a finite EI category. Then the category algebra $k\mathscr{C}$ is Gorenstein if and only if $\mathscr{C}$ is projective over k.*

Proof. Without loss of generality, we assume that $\mathscr{C}$ is skeletal. Otherwise, we take its skeleton $\mathscr{C}_0$, which is equivalent to $\mathscr{C}$. We observe that $\mathscr{C}$ is projective if and only if $\mathscr{C}_0$ is projective and that $k\mathscr{C}$ is Gorenstein if and only if $k\mathscr{C}_0$ is Gorenstein.

Let $\Gamma = \Gamma_{\mathscr{C}}$ be the corresponding upper triangular matrix algebra of $\mathscr{C}$. Observe that $R_i = k\mathrm{Aut}_{\mathscr{C}}(x_i)$ is a group algebra of a finite group. In particular, it is a selfinjective algebra. Then each R_i–R_j-bimodule M_{ij} has finite projective dimension on both sides if and only if it is projective on both sides. Consequently, the statement is immediately due to Proposition 3.4. □

Example 5.2. Let G be a finite group and $\mathcal{P}$ a finite poset. We assume that $\mathcal{P}$ is a *G-poset*, that is, G acts on $\mathcal{P}$ by poset automorphisms. We recall that the *transporter category* $G \propto \mathcal{P}$ is defined as follows. It has the same objects as $\mathcal{P}$, that is, $\mathrm{Obj}(G \propto \mathcal{P}) = \mathrm{Obj}\mathcal{P}$. For $x, y \in \mathrm{Obj}(G \propto \mathcal{P})$, a morphism from x to y is an element g in G satisfying $gx \le y$. The corresponding morphism is denoted by $(g; gx \le y)$. The composition of morphisms is given by the multiplication in G.

We observe that $G \propto \mathcal{P}$ is a finite EI category. Here, we use the fact that $gx \le x$ implies $gx = x$. One can check directly that if $\mathrm{Hom}_{G\propto\mathcal{P}}(x, y) \neq \phi$, then both $\mathrm{Aut}_{G\propto\mathcal{P}}(x)$ and $\mathrm{Aut}_{G\propto\mathcal{P}}(y)$ act freely on $\mathrm{Hom}_{G\propto\mathcal{P}}(x, y)$, in particular, both the left and right stabilizers L_α and R_α of a morphism α are trivial; compare [12, Definition 2.1]. By Corollary 4.4, $G \propto \mathcal{P}$ is projective over k. Then the category algebra $k(G \propto \mathcal{P})$ is Gorenstein by Proposition 5.1; compare [13, Lemma 2.3.2].

Theorem 5.3. *Let k be a field and $\mathscr{C}$ be a finite EI category. Then the category algebra $k\mathscr{C}$ is 1-Gorenstein if and only if $\mathscr{C}$ is a free EI category and projective over k.*

Proof. We assume that $\mathscr{C}$ is skeletal. The reason is similar to the first paragraph in the proof of Theorem 5.1.

Let $\Gamma = \Gamma_{\mathscr{C}}$ be the corresponding upper triangular matrix algebra of $\mathscr{C}$. As mentioned above, each $R_i = k\mathrm{Aut}_{\mathscr{C}}(x_i)$ is a selfinjective algebra. In particular, a module over R_i having finite projective dimension is necessarily projective.

For the "if" part, assume that $\mathscr{C}$ is free and projective over k. Then all bimodules $M_{ij} = k\mathrm{Hom}_{\mathscr{C}}(x_j, x_i)$ are finitely generated and projective on both sides. By Proposition 4.5, each M_t^* is a projective left Γ_t-module for $1 \leq t \leq n-1$. Then Γ is 1-Gorenstein by Proposition 3.8(2).

For the "only if" part, assume that Γ is 1-Gorenstein. Then Proposition 3.8(1) implies that each Γ_t is 1-Gorenstein and each M_t^* is a projective left Γ_t-module for $1 \leq t \leq n-1$. By Corollary 3.6, $\mathrm{pd}_{R_i} M_{i,t+1} < \infty$ for $1 \leq i < t+1 \leq n$, and thus $M_{i,t+1}$ is a projective left R_i-module. By Lemma 3.1, $\mathrm{pd}(M_t^*)_{R_{t+1}} < \infty$ for $1 \leq t \leq n-1$, since M_t^* is a projective left Γ_t-module. Since each M_{ij} is a direct summand of M_{j-1}^* as R_j-modules, we have $\mathrm{pd}(M_{ij})_{R_j} < \infty$ for $1 \leq i < j \leq n$, and thus M_{ij} is a projective right R_j-module. Then the category $\mathscr{C}$ is projective. Since each M_t^* is a projective left Γ_t-module for $1 \leq t \leq n-1$, the category $\mathscr{C}$ is a free EI category by Proposition 4.5. □

Example 5.4. Let $\mathcal{P}$ be a finite poset. For two elements x and y, we write $x < y$ if $x \leq y$ and $x \neq y$. By a *chain*, we mean a totally ordered set. We observe that $\mathcal{P}$ is a free EI category if and only if for any $x \leq y$ in $\mathcal{P}$, the closed interval $[x, y]$ is a chain.

Let G be a finite group and $\mathcal{P}$ a finite G-poset. Consider the transporter category $G \propto \mathcal{P}$. Recall that a morphism $(g; gx \leq y)$ in $G \propto \mathcal{P}$ is an isomorphism if and only if $gx = y$. We observe that a non-isomorphism $(g; gx < y)$ in $G \propto \mathcal{P}$ is unfactorizable if and only if there is no object $z \in \mathrm{Obj}\mathcal{P}$ such that $gx < z < y$. We infer by the UFP that the transporter category $G \propto \mathcal{P}$ is free if and only if the category $\mathcal{P}$ is free. By Example 5.2, we have that the transporter category $G \propto \mathcal{P}$ is projective. Then by Theorem 5.3, the category algebra $k(G \propto \mathcal{P})$ is 1-Gorenstein if and only if the poset $\mathcal{P}$ is free as a category. We mention that this result can be obtained by combining [13, Lemma 2.3.2] and [2, Proposition 2.2].

Example 5.5. (See [8, Theorem 5.3].) Let $\mathscr{C}$ be a finite EI category. Then the category algebra $k\mathscr{C}$ is hereditary if and only if $\mathscr{C}$ is a free EI category satisfying that the endomorphism groups of all objects have orders invertible in k.

We may assume that $\mathscr{C}$ is skeletal. Let $\Gamma = \Gamma_{\mathscr{C}}$ be the corresponding upper triangular matrix algebra of $\mathscr{C}$. We first claim that Γ has finite global dimension if and only if each $\mathrm{Aut}_{\mathscr{C}}(x_i)$ has order invertible in k. In this case, the category $\mathscr{C}$ is projective over k by Corollary 4.4. Indeed, by [4, Corollary 4.21(4)], Γ has finite global dimension if and only if each $R_i = k\mathrm{Aut}_{\mathscr{C}}(x_i)$ has finite global dimension, which is equivalent to that each $R_i = k\mathrm{Aut}_{\mathscr{C}}(x_i)$ is semi-simple. Then we have the claim.

We recall the well-known fact that a finite dimensional algebra is hereditary if and only if it is 1-Gorenstein with finite global dimension. Then the required result follows from the above claim and Theorem 5.3.

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